EECS 461 Short Quiz #3

Probability and Statistics

April 24, 2008

Name: KEY

Closed Book and Closed Notes: Show all work. Provide numerical answers as four-place demimals.

1. (40 %) Independent samples of a random variable X are taken. The sample mean is $\overline{X} = 25.5$ and the true variance is known to be $\sigma^2 = 16$. Consider the following two-sided hypothesis test at the 5% level of significance:

$$H_0: \ \mu = 24,$$
$$H_a: \ \mu \neq 24.$$

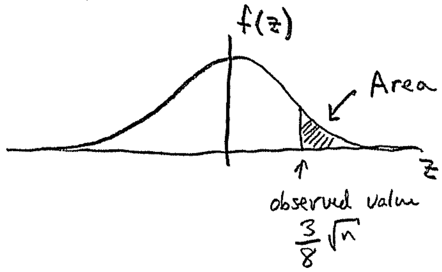
   (a) For $n = 20$, determine the p-value and state on the basis of this p-value whether you accept or reject the null hypothesis H_0.

   (b) For $n = 30$, determine the p-value and state on the basis of this p-value whether you accept or reject the null hypothesis H_0.

   (c) Using p-value calculations, find the maximum value of n for which you accept the null hypothesis H_0.

Because the hypothesis tests the mean, and the variance σ^2 is known, use the normal test statistic. Based on the above data, we observe the value

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{25.5 - 24.0}{4}\sqrt{n} = \frac{3}{8}\sqrt{n}$$



Because the hypothesis is <u>two-sided</u> $p = 2 \times \text{Area}$

(a) $n = 20$ $\quad z = \frac{3}{8}\sqrt{20} = 1.677$

Area $= 0.046766 \leftarrow$ calculator

table + interpolation

Area $= \frac{1}{2} - (0.3 \cdot 0.4525 + 0.7 \cdot 0.4535) \boxed{= 0.0468}$

$p = 2 \times \text{Area} \boxed{= 0.0936}$

not significant enough to reject $\Rightarrow$ $\boxed{\text{Accept } H_0}$

(b) $n = 30$ $\quad z = \frac{3}{8}\sqrt{30} = 2.0540$

Area $= 0.01999$

Area $= \frac{1}{2} - (0.6 \cdot 0.4798 + 0.4 \cdot 0.4803) \boxed{= 0.0200}$

$p = 2 \times \text{Area} = 0.0400$

because $p < \alpha \Rightarrow$ $\boxed{\text{Reject } H_0}$

(c) As long as our observed value of z is less than $z_{\alpha/2}$ we will accept H_0

$$\frac{3}{8}\sqrt{n} < z_{0.025} \Rightarrow n < \left(\frac{8}{3} z_{0.025}\right)^2$$

$z_{0.025} = 1.95996 \leftarrow$ calculator

$z_{0.025} \boxed{= 1.960} \leftarrow$ table

$n < 27.32$

$\boxed{n = 27 \text{ is maximum for accept}}$

2. (30 %) A research engineer for a tire manufacturer is investigating tire life for a new rubber compound and has built 16 tires and tested them to end-of-life in a road test. The sample mean is $\overline{X} = 60139.7$ kilometers and the sample standard deviation is $s = 3645.94$ kilometers.

(a) Can you conclude, using $\alpha = 0.05$, that the standard deviation of tire life is less than 4000 kilometers? State any necessary assumptions about the underlying distribution of the data.

(b) Find the p-value for the test in part (a).

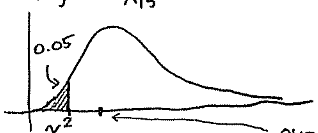
(a) Hypothesis test involves variance, use χ^2 test

$H_0: \sigma^2 = (4000)^2$

$H_a: \sigma^2 < (4000)^2$

$\boxed{\text{Assume samples in } X \text{ are normal}}$

$f(x)$ for χ^2_{15}



$\chi^2_{15;0.95}$
$= 7.261 \leftarrow$ table
$= 7.26094 \leftarrow$ calculator

we observe the variable

$$\chi^2_{15} = \frac{(n-1)s^2}{\sigma^2} = \frac{15\,(3645.94)^2}{(4000)^2} = 12.4621$$

Formulating the question in terms of a confidence interval

$$\underbrace{P\left(\chi^2_{15;0.95} < \frac{(n-1)s^2}{\sigma^2}\right)}_{\text{solve for } \sigma^2} \Rightarrow \sigma^2 \overset{?}{<} \frac{(n-1)s^2}{\chi^2_{15;0.95}}$$

$$(4000)^2 \overset{?}{<} \frac{15\,(3645.94)^2}{7.261}$$

$$16{,}000{,}000 \overset{\checkmark}{<} 27{,}461{,}000$$

$\boxed{\text{Accept } H_0}$

(b) The p-value is the area to the left of our observed value of $\chi^2_{15} = 12.4621$ (this is a one-sided hypothesis, so we do <u>not</u> double the area). The table is no good here, because p is large $\Rightarrow p = 0.3562 \leftarrow$ calculator.

Because $p > \alpha$ $\boxed{\text{Accept } H_0}$

3. (30 %) Let $X_1, X_2, \cdots, X_7$ denote a random sample from a population having mean μ and variance σ^2. Consider the following estimators of μ: $\hat{\mu}_1 = \dfrac{X_1 + X_2 + \cdots + X_7}{7}$, and $\hat{\mu}_2 = \dfrac{2X_1 - X_6 + X_4}{2}$.

(a) What is the expected value of $\hat{\mu}_1$?

(b) What is the expected value of $\hat{\mu}_2$?

(c) What is the variance of $\hat{\mu}_1$?

(d) What is the variance of $\hat{\mu}_2$?

(e) Which estimator is "better"? In what sense is it better?

(a) $E(\hat{\mu}_1) = \frac{1}{7}\left(E(X_1) + E(X_2) + \cdots + E(X_7)\right) = \frac{1}{7}(7\mu) \boxed{= \mu}$

(b) $E(\hat{\mu}_2) = \frac{1}{2}\left(2E(X_1) - E(X_6) + E(X_4)\right) = \frac{1}{2}(2\mu - \mu + \mu) \boxed{= \mu}$

Both estimators are <u>unbiased</u> because their expected values are μ

(c) $Var(\hat{\mu}_1) = \frac{1}{7^2}\left(Var(X_1) + Var(X_2) + \cdots + Var(X_7)\right) = \frac{1}{49}(7\sigma^2) \boxed{= \frac{1}{7}\sigma^2}$

(d) $Var(\hat{\mu}_2) = \frac{1}{2^2}\left(2^2 Var(X_1) + (-1)^2 Var(X_6) + Var(X_4)\right) = \frac{1}{4}(4\sigma^2 + \sigma^2 + \sigma^2) \boxed{= \frac{3}{2}\sigma^2}$

(e) both estimators are unbiased, but $Var(\hat{\mu}_1) < Var(\hat{\mu}_2)$.

$\boxed{\hat{\mu}_1 \text{ is better than } \hat{\mu}_2 \text{ because it has smaller variance}}$

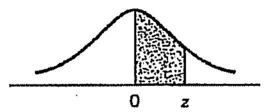
Description	Formula
Sample Variance	$s^2 \quad = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\left(\overline{X}\right)^2}{n-1}$
Normal test statistic	$z \quad = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ ← Problem 1
t test statistic	$t_{n-1} \quad = \frac{\overline{X}-\mu}{s/\sqrt{n}}$
Chi-squared test statistic	$\chi^2_{n-1} \quad = \frac{(n-1)s^2}{\sigma^2}$ ← Problem 2

Standard Normal Distribution

The entries in this table give the areas under the standard normal curve from 0 to *z*.

Problem 1(c)

$z_{0.025} = 1.96$

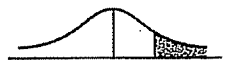


z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	,4532 .4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	0.7 .4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878 0.4	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Problem 1(b) Problem 1(a)

The t Distribution Table

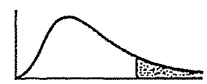
The entries in the table give the critical values of t for the specified number of degrees of freedom and areas in the right tail.



	Area in the Right Tail under the t Distribution Curve					
df	**.10**	**.05**	**.025**	**.01**	**.005**	**.001**
1	3.078	6.314	12.706	31.821	63.657	318.309
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733

Chi-Squared Distribution Table

The entries in this table give the critical values of χ^2 for the specified number of degrees of freedom and areas in the right tail.



	Area in the Right Tail under the Chi-square Distribution Curve									
df	**.995**	**.990**	**.975**	**.950**	**.900**	**.100**	**.050**	**.025**	**.010**	**.005**
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801

Problem 2