EECS 461 Midterm Exam

Department of Electrical Engineering and Computer Science
University of Kansas
March 13, 2008
Instructor: Erik Perrins

Problem	Points	Score
1	15	
2	25	
3	20	
4	20	
5	20	
Total:	100	

The following rules apply for this exam:

1. Closed book and closed notes. A calculator is required for full credit.
2. Provide numerical answers as four-place demimals.
3. The exam must be completed within the class period (75 minutes).

REMEMBER:

- **Show all your work!**
- The exam is **double-sided**.
- Helpful formulas and tables are found in the back of the exam.
- If you can't finish a problem, then at least **set it up**.
- Be neat, write legibly.

Name: KEY

1. [15 points] Fifty-two percent of the students at a certain college are females. Five percent of the students in this college are majoring in computer science. Two percent of the students are women majoring in computer science. If a student is selected at random,

   (a) find the probability that this student is female, given that the student is majoring in computer science;

   (b) find the probability that this student is majoring in computer science, given that the student is female.

Let A be the event that the student is female

Let B be the event that the student is majoring in computer science

(a) $$P(A|B) = \underbrace{\frac{P(A \cap B)}{P(B)}}_{\text{Definition of Conditional Probability}} = \frac{0.02}{0.05} \boxed{= 0.4000}$$

(b) $$P(B|A) = \underbrace{\frac{P(A \cap B)}{P(A)}}_{\text{Definition of conditional Probability}} = \frac{0.02}{0.52} \boxed{= 0.0385}$$

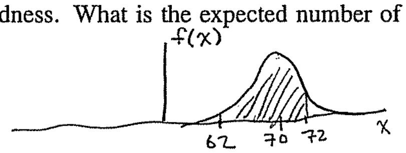
2. [25 points] The Rockwell hardness of a particular alloy is normally distributed with a mean of 70 and a variance of 16.

   (a) A specimen is acceptable only if its hardness is between 62 and 72. What is the probability that a randomly chosen specimen has an acceptable hardness?

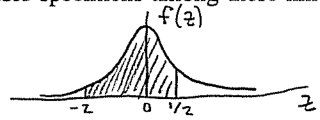
   (b) If the acceptable range of hardness was between $(70 - a)$ and $(70 + a)$, for what value of a would 95% of all specimens have acceptable hardness?

   (c) Going back to the acceptable range between 62 and 72, we randomly select nine specimens and independently determine their hardness. What is the expected number of acceptable specimens among these nine specimens?

(a) $X \sim N(70, 4)$



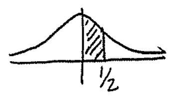
$\Rightarrow$



$$P(62 < X < 72) = P\left(\frac{62-70}{4} < \underbrace{\frac{X-70}{4}}_{Z} < \frac{72-70}{4}\right)$$

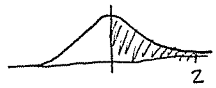
$$= P\left(-2 < Z < \frac{1}{2}\right) \boxed{= 0.6687}$$ ← via calculator

Via the table: $Z = \frac{1}{2} \Rightarrow \text{Area} = 0.1915$



$0.1915 + 0.4772 \boxed{= 0.6687}$

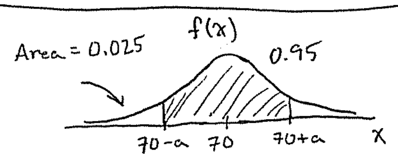
$Z = 2 \Rightarrow \text{Area} = 0.4772$



no interpolation necessary since $\frac{1}{2}$ and 2 are exactly in the table.

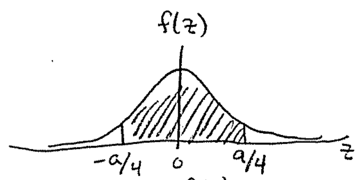
(b) Find a such that

$$P(70 - a < X < 70 + a) = 0.95$$



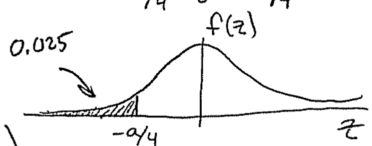
$$P\left(\frac{70-a-70}{4} < \frac{X-70}{4} < \frac{70+a-70}{4}\right) = 0.95$$

$$P\left(-\frac{a}{4} < Z < \frac{a}{4}\right) = 0.95$$



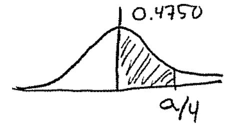
For my calculator, I find

$$CDF_Z\left(\frac{-a}{4}\right) = 0.025$$



$$-\frac{a}{4} = CDF_Z^{-1}(0.025) \Rightarrow a = -4 \cdot (-1.95996)$$
$$\boxed{= 7.8399}$$

With the table, I find the entry with an area of 0.4750



and that is $a/4 \Rightarrow a = 4(1.96) \boxed{= 7.8400}$

(c) This experiment involves 9 Bernoulli trials, each with $p = 0.6687$, and the counting of success/failure out of 9 trials is a Binomial RV

$$E\{\text{Binomial RV}\} = np = 9(0.6687)$$
$$\boxed{= 6.0184}$$

3. [20 points] The manufacturing of semiconductor chips produces 2% defective chips. Assume the chips are independent and that a lot contains 100 chips.

(a) Using the binomial distribution, compute the probability that more than 2 chips in the lot are defective.

(b) Using the Poisson approximation to the binomial distribution, approximate the probability that less than 3 chips in the lot are defective.

(c) Using the normal approximation to the binomial distribution, approximate the probability that between 2 and 3 chips in the lot are defective.

(a) When $X \sim \text{Binomial}(n=100, p=0.02)$

$$P(3 \le X \le 100) = 1 - P(0 \le X \le 2) = 1 - \sum_{x=0}^{2} \binom{100}{x} (0.02)^x (0.98)^{100-x} \boxed{= 0.3233}$$

(b) When X is approximately distributed as Poisson $(\lambda = np = 2)$

$$P(0 \le X \le 2) = \sum_{x=0}^{2} \frac{e^{-2} 2^x}{x!} \boxed{= 0.6767}$$

(with more decimal places, we get 0.676676, which is very close to the exact binomial result of 0.676686)

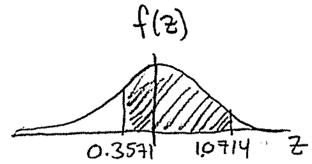
(c) When X is approximately distributed as $N(\mu = np = 2,\ \sigma = \sqrt{npq} = \sqrt{100 \cdot 0.02 \cdot 0.98} = 1.4)$

$$P(1.5 < X < 3.5) = F(3.5) - F(1.5)$$

calculator $\longrightarrow$ $\boxed{= 0.4975}$

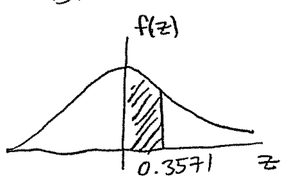
(compare with the exact binomial value $P(2 \le X \le 3) = \sum_{x=2}^{3} \binom{100}{x} (0.02)^x (0.98)^{100-x} = 0.4557$)

Using the standard normal table



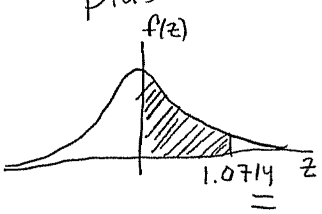
$$P\left(\frac{1.5-2}{1.4} < \frac{X-\mu}{\sigma} < \frac{3.5-2}{1.4}\right) = P(-0.3571 < Z < 1.0714)$$

Break this into



$$\longrightarrow \underbrace{(0.71)}_{\text{Interpolation weight}} \underbrace{(0.1406)}_{\text{Table entry for } z=0.36} + \underbrace{(1-0.71)}_{\text{Interpolation weight}} \underbrace{(0.1368)}_{\text{Table entry for } z=0.35} = 0.1395$$

plus



$$\longrightarrow \underbrace{(0.14)}_{\text{interpolation weight}} \underbrace{(0.3599)}_{\text{Table entry for } z=1.08} + \underbrace{(1-0.14)}_{\text{interpolation weight}} \underbrace{(0.3577)}_{\text{Table entry for } z=1.07} = 0.3580$$

Final answer: $0.1395 + 0.3580 \boxed{= 0.4975}$

4. [20 points] The time between arrivals of taxis at a busy intersection is exponentially distributed with a mean of 10 minutes.

   (a) What is the probability that you will wait longer than one hour for a taxi?

   (b) Suppose you have already been waiting for one hour for a taxi, what is the probability that one arrives within the next 10 minutes?

   (c) Determine x such that the probability that you wait less than x minutes is 0.10.

Mean of exponential is $\frac{1}{\lambda} = 10 \Rightarrow \lambda = 0.10$

(a) $P(60 \text{ minutes} < X < \infty) = F(\infty) - F(60)$

$$= \left[1 - e^{-0.10(\infty)}\right] - \left[1 - e^{-0.10(60)}\right]$$

$$= 1 - 0.9975 \boxed{= 0.0025}$$

$$F(x) = \int_0^x \lambda e^{-\lambda u}\, du = 1 - e^{-\lambda x}$$

(b) Because the exponential distribution is "memoryless"

$$P(60 < X < 70 \mid X > 60) = P(0 < X < 10)$$

$$= F(10) - F(0) \boxed{= 0.6321}$$

(c) Find x such that

$$P(0 < X < x) = 0.10 = F(x) - \cancel{F(0)}^{\,0}$$

, therefore $0.10 = F(x)$

$$= 1 - e^{-0.1x}$$

$$\Rightarrow e^{-0.1x} = 0.9$$

$$-0.1x = \ln(0.9)$$

$$\Rightarrow x = -10\ln(0.9) \boxed{\begin{aligned} &= 1.0536 \\ &= 1 \text{ minute, } 3.22 \text{ seconds} \end{aligned}}$$

5. [20 points] Suppose that X is a random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of C.

(b) Determine the cumulative distribution function of X.

(c) Compute the mean of X.

(d) Compute the variance of X.

(e) Find $P(X > 1)$.

(a) We know the area of the PDF must equal 1

$$1 = \int_0^2 C(4x-2x^2)\,dx = C\left[2x^2 - \tfrac{2}{3}x^3\right]_{x=0}^{2} = C\left[2(4) - \tfrac{2}{3}(8)\right] = C\tfrac{8}{3} \Rightarrow \boxed{C = \tfrac{3}{8}}$$

(b) $$F(x) = \int_0^x \tfrac{3}{8}(4u - \tfrac{2}{3}u^2)\,du = \tfrac{3}{8}\left[2u^2 - \tfrac{2}{3}u^3\right]_{u=0}^{x} = \tfrac{3}{4}x^2 - \tfrac{1}{4}x^3, \text{ when } 0 < x < 2$$

$$\boxed{F(x) = \begin{cases} 1 & x > 2 \\ \tfrac{3}{4}x^2 - \tfrac{1}{4}x^3 & 0 < x < 2 \\ 0 & x < 0 \end{cases}}$$

(c) $$\mu = E(X) = \int_0^2 \tfrac{3}{8}(4x^2 - 2x^3)\,dx = \tfrac{3}{8}\left[\tfrac{4}{3}x^3 - \tfrac{1}{2}x^4\right]_{x=0}^{2} = \tfrac{3}{8}\left(\tfrac{4}{3}(8) - \tfrac{1}{2}(16)\right) \boxed{= 1}$$

(d) First find $E(X^2)$, then use $\sigma^2 = E(X^2) - \mu^2$

$$E(x^2) = \int_0^2 \tfrac{3}{8}(4x^3 - 2x^4)\,dx = \tfrac{3}{8}\left[x^4 - \tfrac{2}{5}x^5\right]_{x=0}^{2} = \tfrac{3}{8}\left(16 - \tfrac{2}{5}(32)\right) = \tfrac{6}{5}$$

$$\sigma^2 = \tfrac{6}{5} - 1^2 \boxed{= \tfrac{1}{5}}$$

(e) $$P(1 \le X \le \infty) = F(\infty) - F(1) = 1 - \left(\tfrac{3}{4} - \tfrac{1}{4}\right) \boxed{= \tfrac{1}{2}}$$

(yes, I know, these are not 4-place decimals)