EECS 461 Short Quiz #2

Probability and Statistics

March 5, 2007

Name: KEY

Closed Book and Closed Notes: Show all work. Provide numerical answers as four-place demimals.

1. (50 %) You work at the local hospital and have two possible routes you can take to work. You are a law-abiding citizen who *always* observes the speed limit. Here is a description of the two routes:

   - **Route A** is 15 km long, is free of stops, and has a speed limit of 60 km/h.
   - **Route B** is shorter at 6 km long, but has *12 traffic lights* and a speed limit of only 40 km/h. Each light is red with a probability of 0.3 and operates independently of the others. You always wait exactly 2 minutes at each red light, after which you quickly accelerate to the posted speed limit.

   a. Which route will give you the shortest expected driving time?

We should be able to write the driving time of each route using our common sense (distance = rate × time)

$$T_A = \frac{15\text{ km}}{60\text{ km/hr}} = 0.25\text{ hr} \boxed{= 15\text{ minutes}}$$ ← nothing random in this equation

$$T_B = \frac{6\text{ km}}{40\text{ km/hr}} + (2\text{ min})\cdot X = 9\text{ min} + (2\text{ min})X$$

where X is the number of red lights we encounter.

This whole chapter deals with Bernoulli trials, which can be modeled as a "loaded coin toss." Can we model the red/green traffic lights as a loaded coin toss? Yes! And X counts the number of reds (successes) out of 12 independent trials, so X is binomial with $p = 0.3$ and $n = 12$

$$E(T_B) = E(9 + 2X) = 9 + 2\cancel{E(X)}^{np} = 9 + 2\cdot 12\cdot 0.3 \boxed{= 16.2\text{ minutes}}$$

$\boxed{\text{Take route A}}$

   b. One morning your neighbor's wife, who is expecting a baby, tells you she needs to get to the hospital in 11 minutes or less. What is the probability of this happening if you take Route A? What about Route B?

We are looking for $P(T_A \le 11\text{ minutes})$, but T_A is a constant so this probability is $\boxed{\text{zero!}}$

What about $P(T_B \le 11\text{ minutes})$?

$$= P(9 + 2X \le 11)$$
$$= P(2X \le 2)$$
$$= P(X \le 1) = \sum_{x=0}^{1} \binom{12}{x}(0.3)^x(0.7)^{12-x} \boxed{= 0.0850}$$

You have no chance of making it with Route A, and a slim chance of making it with Route B. Get going!

   c. For some reason, you wish the two routes had the same expected driving time. How many traffic lights must you construct or dismantle along Route B to accomplish this?

We want $E(T_B) = T_A = 15$ minutes

$$E(T_B) = \underbrace{9 + 2\cdot n\cdot p}_{\text{from part a}} = 15\text{ minutes}$$, solve for n

$\Rightarrow \boxed{n = 10}$, but there are 12 lights right now, so

$\boxed{\text{dismantle 2 lights}}$

2. (25 %) The number of cracks in a section of interstate highway that are significant enough to require repair is assumed to follow the Poisson distribution with a mean of two cracks per km.

a. What is the probability that there are no cracks that require repair in 5 km of highway?

This problem is very similar to Example 2.14.1, which we covered in class. In this example, we learned that the Poisson parameter λ can be adjusted to have the proper units. We are given $\lambda = 2$ cracks/km, but we need it in terms of 5 km, so we convert to $\lambda = 10$ cracks / 5 km

$$P(\text{no cracks in 5 km}) = \underset{\substack{\text{Poisson RV} \\ \text{with } \lambda = 10}}{P(0 \le X \le 0)} = \sum_{x=0}^{0} \frac{e^{-10} 10^x}{x!} \boxed{= 4.540 \times 10^{-5}}$$

b. What is the probability that at least one crack requires repair in 1/2 km of highway?

Now we need to convert our units to $\lambda = \frac{1 \text{ crack}}{1/2 \text{ km}}$

$$P(1 \le X \le \infty) = 1 - P(0 \le X \le 0)$$

Poisson RV with $\lambda = 1$

$$= 1 - \sum_{x=0}^{0} \frac{e^{-1} 1^x}{x!} \boxed{= 0.6321}$$

3. (25 %) Argue that for any random variable X, the following inequality holds:

$$E(X^2) \ge [E(X)]^2.$$

When do we achieve equality?

The only place where we have seen these terms together is

$$\sigma^2 = E(X^2) - [E(X)]^2,$$

if the variance was never negative, then the inequality above would be true:

$$0 \le \sigma^2 = E(X^2) - [E(X)]^2 \Rightarrow E(X^2) \ge [E(X)]^2$$

thus this is true

So, can the variance be negative?

$$\sigma^2 = E((X-\mu)^2) = \sum_x \underbrace{(x-\mu)^2}_{\text{never negative}} \underbrace{P(X=x)}_{\text{never negative}} \quad \therefore$$

variance is never negative!

If $E(X^2) = [E(X)]^2$, it can only mean that $\sigma^2 = 0$

$$\sigma^2 = \sum_x \underbrace{(x-\mu)^2}_{\substack{\text{this must} \\ \text{be zero} \\ \text{always}}} \underbrace{P(X=x)}_{\substack{\text{can't be zero} \\ \text{everywhere, must} \\ \text{add up to 1}}}$$

when is $X - \mu = 0$?

when $X = \mu$, or

$\boxed{\text{when } X \text{ is a constant}}$