EECS 461 Midterm Exam

Department of Electrical Engineering and Computer Science
University of Kansas
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[Version A] Instructor: Erik Perrins
[Version B] Instructor: Erik S. Perrins

Look at your exam to see which version you took.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

The following rules apply for this exam:

1. Closed book and closed notes. A calculator is required for full credit.
2. Provide numerical answers as four-place demimals.
3. The exam must be completed within the class period (75 min.).

REMEMBER:

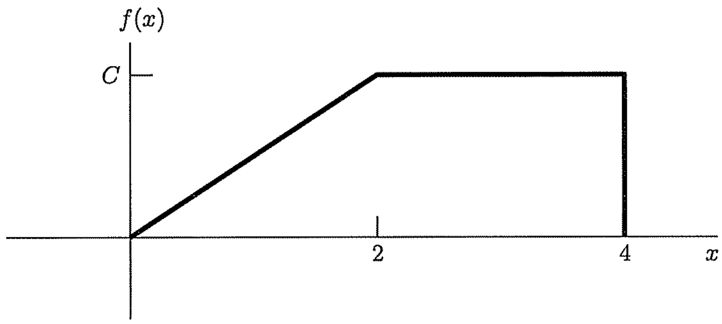
- Show all your work!
- The exam is **double-sided**.
- Helpful formulas and tables are found in the back of the exam.
- If you can't finish a problem, then at least **set it up**.
- Be neat, write legibly.

Name: KEY

1. [20 points] The probability density function (PDF) of the random variable X is defined as

$$f(x) = \begin{cases} \frac{1}{2}Cx, & 0 \le x < 2 \\ C, & 2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

and is graphed below.



(a) Determine the value of C so that $f(x)$ is a proper PDF.

(b) Compute the mean of X.

(c) Compute the variance of X.

(d) Compute $P(1.0 \le X \le 1.5)$.

(a) Area under the curve $= \underbrace{\frac{1}{2}(2)(C)}_{\text{triangle}} + \underbrace{(2)(C)}_{\text{rectangle}} = C + 2C = 3C = \underbrace{1}_{\text{because its a pdf}}$ $\Rightarrow \boxed{C = 1/3}$

(b) $E(X) = \int_0^4 x f(x)dx = \int_0^2 x(\frac{1}{6}x)dx + \int_2^4 x(\frac{1}{3})dx = \frac{1}{18}x^3\Big|_0^2 + \frac{1}{6}x^2\Big|_2^4 \boxed{= 2.4444}$

(c) Use $\sigma^2 = E(X^2) - [E(X)]^2$

$E(X^2) = \int_0^4 x^2 f(x)dx = \int_0^2 x^2(\frac{1}{6}x)dx + \int_2^4 x^2(\frac{1}{3})dx = \frac{1}{24}x^4\Big|_0^2 + \frac{1}{9}x^3\Big|_2^4 = 6.8888$

$\sigma^2 = 6.8888 - (2.4444)^2 \boxed{= 0.9136}$

(d) $P(1.0 \le X \le 1.5) = \int_{1.0}^{1.5} f(x)dx = \int_{1.0}^{1.5} (\frac{1}{6}x)dx = \frac{1}{12}x^2\Big|_{1.0}^{1.5} \boxed{= 0.1042}$

2. [20 points] The termination of a chemical reaction occurs at a random time T between 6 and 7.5 hours after the start of the experiment. The time follows a **uniform** distribution.

   (a) What is the probability that the reaction lasts at least 6.5 hours and no more than 6.75 hours?

   (b) **[Version A]** If the reaction is run four independent times, what is the probability that in **at most one** of the four replications of the experiment the reaction will last no more than 6.5 hours?
   [Version B] If the reaction is run four independent times, what is the probability that in **at least two** of the four replications of the experiment the reaction will last no more than 6.5 hours?

(a) What is the probability that $6.5 \le T \le 6.75$?

$T \sim \underbrace{U(6.0, 7.5)}$

$$P(6.5 \le T \le 6.75) = \int_{6.5}^{6.75} f(t)\,dt$$

from the formula sheet we have

$$f(t) = \begin{cases} \frac{1}{7.5-6.0} & 6.0 \le t \le 7.5 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{1.5} & 6.0 \le t \le 7.5 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{6.5}^{6.75} \left(\frac{1}{1.5}\right) dt$$

$$= \frac{t}{1.5}\Big|_{6.5}^{6.75}$$

$$\boxed{= 0.1667}$$

(b) In each trial, the reaction either lasts longer than 6.5 hours or it doesn't. This can be modeled as a "loaded coin toss". Therefore, we are dealing with Bernoulli trials, and we are asked to find the number of "Successes" in $n = 4$ trials, this is a Binomial distribution problem

$$P(\text{Success}) = \underbrace{p}_{\text{parameter in Binomial PDF}} = P(T \leqslant 6.5) = \int_{6.0}^{6.5} \left(\frac{1}{1.5}\right) dt = \frac{t}{1.5}\Big|_{6.0}^{6.5} \boxed{= \frac{1}{3} = 0.3333}$$

[Version A]

$$P(\underbrace{0 \le X \le 1}_{\text{Binomial RV with } n=4 \text{ and } p=1/3}) = \sum_{x=0}^{1} \binom{4}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$$

$$\boxed{= 0.5926}$$

[Version B]

$$P(\underbrace{2 \le X \le 4}_{\text{Binomial RV with } n=4 \text{ and } p=1/3}) = \sum_{x=2}^{4} \binom{4}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$$

$$\boxed{= 0.4074}$$

3. [20 points] An **exponential** process has a reliability of $R = 0.85$ at a time of $t = 200$ hours. In other words, $R(200) = 0.85$.

   (a) What is the expected lifetime of the process before it fails?

   (b) At what time does the process have $R = 0.95$?

   (c) **[Version A]** If the process has operated without failure for **1000** hours, what is the probability it will fail in the next **600** hours?
   [Version B] If the process has operated without failure for **600** hours, what is the probability it will fail in the next **1000** hours?

(a) From the formula sheet we have $R(t) = 1 - F(t)$ and from memory, or by integrating the exponential PDF on the formula sheet we have $R(t) = e^{-\lambda t}$

$R(200) = e^{-\lambda 200} = 0.85 \Rightarrow -\lambda 200 = \ln(0.85) \Rightarrow \lambda = 8.126 \times 10^{-4}$

From the formula sheet we have $\mu = \frac{1}{\lambda} \boxed{= 1231 \text{ hours}}$

(b) $R(t) = e^{-\lambda t} = 0.95$ ← solve for t

$-\lambda t = \ln(0.95) \Rightarrow t = \frac{\ln(0.95)}{-\lambda} \boxed{= 63.12 \text{ hours}}$

(c) This is just like our first encounter with the exponential distribution on page 196

If we are given that the process has operated for a given amount of time, all that does is reset the clock back to zero

[Version A]

$P(1000 \le T \le 1600 \mid T > 1000)$

$= P(0 \le T \le 600)$ ← memoryless property

$= F(600)$

$= 1 - e^{-\lambda 600}$

$\boxed{= 0.3859}$

[Version B]

$P(600 \le T \le 1600 \mid T > 600)$

$= P(0 \le T \le 1000)$ ← memoryless property

$= F(1000)$

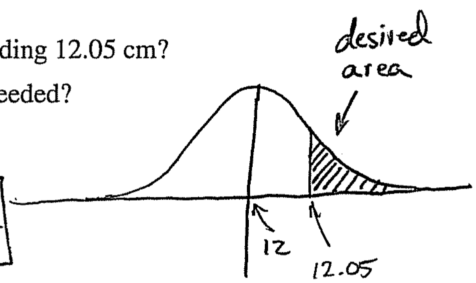
$= 1 - e^{-\lambda 1000}$

$\boxed{= 0.5563}$

4. [20 points] The inside diameter of a piston ring is **normally** distributed with a mean of 12 cm and a **standard deviation** of 0.02 cm.

(a) What is the probability that a piston ring will have a diameter exceeding 12.05 cm?

(b) What inside diameter value x has a probability of 0.10 of being exceeded?



(a) $X \sim N(12, 0.02)$

$P(X > 12.05)$ $\boxed{= 0.0062 \text{ using fancy calculator}}$

or $= P(X - 12 > 12.05 - 12)$

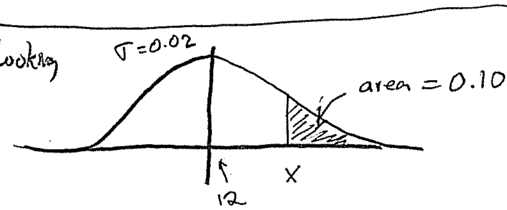
$= P\left(\frac{X-12}{0.02} > \frac{12.05-12}{0.02}\right)$

$= P(Z > z)$ where $z = 2.50$, from the table we get 0.4938, but we want the tail probability, so

$= 0.5 - 0.4938$

$\boxed{= 0.0062}$

This is ultimately what we are looking for. We want the X that corresponds to a tail probability of 0.10.



a tail probability of 0.10 corresponds to a table probability of 0.4000

So, read the table backwards and find the z value that corresponds to a table probability of 0.40 (See the table)

this value of z is ≈ 1.28

using our standard approach we have

$\frac{X-\mu}{\sigma} = z \approx 1.28$ ← solve for X

$\Rightarrow \boxed{X = 12.0256}$

using my more accurate value of z I get $\boxed{X = 12.02563\ldots}$

5. [20 points] This semester we have talked about the fact that surveys are often conducted to estimate various things. Sometimes the information we are seeking in the survey is *sensitive*. For example, suppose we want to survey 1000 University of Kansas students to find out what fraction of them have used *marijuana* in the past month. If we just ask the direct question "Have you used marijuana in the past month, yes or no?," some of them might hesitate to answer truthfully, thus affecting the accuracy of our survey.

   Consider this alternative approach. Suppose we have a stack of 1000 cards that are randomly shuffled. On each card is written **one** of the following **two** questions:

   - **Question A:** Have you used marijuana in the past month, yes or no?
   - **Question B:** Have you been 100% marijuana-free in the past month, yes or no?

   The person being surveyed (the KU student) draws a random card and answers the question. Since the question is "blind" to the person conducting the survey (you), the KU student feels comfortable answering the question truthfully. The probability distribution of the stack of cards is:

   - $P(A) = 0.3$
   - $P(B) = 1 - P(A) = 0.7$

   We conduct the survey as outlined above and we get **[Version A = 612] [Version B = 604]** "Yes" answers. Based on these responses and the format of the survey, is it possible to estimate the probability that a student uses marijuana, $P(U)$? If so, what is your estimate of $P(U)$?

   [Hint: You should use conditional probability concepts from Chapter 1 to solve this problem.]

Using the law of total probability from Chapter 1
(see denominator of Bayes' Rule on formula sheet)

Let the event Y = the student answers "Yes"

we are given $P(Y)$ in the problem. We also recognize that
U = the student uses marijuana and
$\bar{U}$ = " " does not use "
form a partition on the probability space

this is $1-P(U)$

$$P(Y) = \underbrace{P(Y|U)}_{\text{If they use, they will answer Yes if they get Question A, so this is } P(A)}P(U) + \underbrace{P(Y|\bar{U})}_{\text{By similar logic this is } P(B)}P(\bar{U})$$

$$P(Y) = P(A)P(U) + P(B)(1-P(U))$$

$$\Rightarrow P(U) = \frac{P(Y) - P(B)}{P(A) - P(B)}$$

[Version A]

$$P(U) = \frac{\frac{612}{1000} - 0.7}{0.3 - 0.7} \boxed{= 0.2200}$$

[Version B]

$$P(U) = \frac{\frac{604}{1000} - 0.7}{0.3 - 0.7} \boxed{= 0.2400}$$

Helpful formulas:

Binomial Distribution: $P(X=x)=\binom{n}{x}p^x q^{n-x}, \qquad x=0,1,2,\cdots,n$

$$\mu=np, \qquad \sigma^2=npq$$

Geometric Distribution: $P(X=x)=q^{x-1}p, \qquad x=1,2,3,\cdots$

$$\mu=\frac{1}{p}, \qquad \sigma^2=\frac{q}{p^2}$$

Negative Binomial Distribution: $P(X=x)=\binom{x-1}{r-1}p^{r-1}q^{x-r}p, \qquad x=r,r+1,r+2,\cdots$

$$\mu=\frac{r}{p}, \qquad \sigma^2=\frac{r\cdot q}{p^2}$$

Hypergeometric Distribution: $P(X=x)=\dfrac{\binom{D}{x}\cdot\binom{N-D}{n-x}}{\binom{N}{n}}, \qquad x=0,1,2,\cdots,\min(n,D)$

$$\mu=n\cdot\frac{D}{N}, \qquad \sigma^2=n\cdot\frac{D}{N}\cdot\frac{N-D}{N}\cdot\frac{N-n}{N-1}$$

Poisson Distribution: $P(X=x)=\dfrac{e^{-\lambda}\cdot\lambda^x}{x!}, \qquad x=0,1,2,\cdots$

$$\mu=\lambda, \qquad \sigma^2=\lambda$$

Uniform Distribution: $\boxed{f(x)=\begin{cases}\dfrac{1}{b-a}, & a\le x\le b\\ 0, & \text{otherwise}\end{cases}}$ ← use on Problem #2

$$\mu=\frac{b+a}{2}, \qquad \sigma^2=\frac{(b-a)^2}{12}$$

Exponential Distribution (we assume $a=0$): $f(x)=\lambda e^{-\lambda x}, \qquad x\ge 0$

$$\mu=\frac{1}{\lambda}, \qquad \sigma^2=\frac{1}{\lambda^2}$$

Normal (Gaussian) Distribution: $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Reliability: $\boxed{R(t)}=P(T>t)=1-P(T<t)=\boxed{1-F(t)}$ ← use on Problem #3

Hazard Rate: $h(t)=-\dfrac{R'(t)}{R(t)}$ as $\Delta t\to 0$

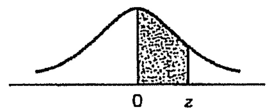
Bayes' Rule: $P(A_i|B)=\dfrac{P(B|A_i)P(A_i)}{P(B)}=\dfrac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n}P(B|A_j)P(A_j)}$

$\boxed{P(B)=\sum_{j=1}^{n}P(B|A_j)P(A_j)}$

Law of total probability
use on Problem #5

Standard Normal Distribution

The entries in this table give the areas under the standard normal curve from 0 to *z*.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	$\boxed{.4938}$	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Prob #4a

Problem #4 b

this one is the closest $\Rightarrow z = 1.28$

I will give you full credit for this one also $\Rightarrow z = 1.29$

using my fancy calculator I get $z = 1.2815515655$, but who's counting