Suppose that $X \sim N(\mu, \sigma)$, and let $Y = e^X$

a) Find the mean and variance of Y.

$$E(Y) = \int_{-\infty}^{\infty} (e^x)\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx$$

Let's work with the exponents inside the integral

$$x - \frac{(x-\mu)^2}{2\sigma^2} = \frac{-1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2 - 2\sigma^2 x\right)$$ ← complete the square

$$= -\frac{1}{2\sigma^2}\left(x^2 - 2(\mu+\sigma^2)x + (\mu+\sigma^2)^2 - 2\mu\sigma^2 - \sigma^4\right)$$

$$= \frac{-1}{2\sigma^2}\left(x - (\mu+\sigma^2)\right)^2 + \mu + \frac{\sigma^2}{2}$$, now return to the integral

$$E(Y) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}} dx}_{\text{integrates to } 1} \, e^{\mu+\frac{\sigma^2}{2}} \boxed{= e^{\mu+\frac{\sigma^2}{2}}}$$

To find the varience, first find

$$E(Y^2) = \int_{-\infty}^{\infty} (e^{2x})\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right) dx$$

by similar arguments the final exponent is

$$\frac{-1}{2\sigma^2}\left(x - (\mu+2\sigma^2)\right)^2 + 2\mu + 2\sigma^2$$ which gives

$$E(Y^2) = e^{2\mu+2\sigma^2}$$

$$Var(Y^2) = E(Y^2) - [E(Y)]^2$$

$$= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$$

$$\boxed{= e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)}$$

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b) Find the PDF of Y

Notation: X has PDF $f(x)$ and CDF $F(x)$

Y has PDF $g(y)$ and CDF $G(y)$

Following Example 4.31, we first find $G(y)$

$$G(y) = P(Y < y) = P(e^X < y)$$

we want to solve for x, but we have to ask ourselves if taking the $\ln(\cdot)$ of both sides will mess up the inequality, the answer is "no", if $e^X < y$, then $X < \ln(y)$ because $\ln(\cdot)$ is monotonic

$$= P(X < \ln(y))$$

$$= F(\ln(y))$$

Now we use the fact that

$$f(y) = \frac{d}{dy}[G(y)]$$

$$= \frac{d}{dy}\left[F(\ln(y))\right]$$

use chain rule, just like taking the derivative of $\cos(x^2)$

$$= f(\ln(y))\frac{d}{dy}\left[\ln(y)\right]$$

$$= f(\ln(y))\frac{1}{y}$$

$$\boxed{= \frac{1}{\sqrt{2\pi\sigma^2}}\frac{1}{y}e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}, \quad y > 0}$$