is that of a normal distribution with mean 0 and variance 1; that is, $Z \sim N(0, 1)$, and we say that Z has a *standard normal distribution.* A graph of the probability density function is shown in Fig. 7-2. The corresponding distribution function is Φ, where

CDF of the standard Normal

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du, \tag{7-8}$$

and this function has been well tabulated. A table of the integral in equation 7-8 has been provided in Table II of the Appendix. In fact, many software packages such as Excel and Minitab provide functions to evaluate $\Phi(z)$. For instance, the Excel call NORMSDIST(z) does just this task. As an example, we find that NORMSDIST(1.96) = 0.9750. The Excel function NORMSINV returns the inverse CDF. For example, NORMSINV(0.975) = 1.960. The functions NORMDIST(x, μ, σ, TRUE) and NORMINV give the CDF and inverse CDF of the $N(\mu, \sigma^2)$ distribution.

7-2.5 Problem-Solving Procedure

The procedure for solving practical problems involving the evaluation of cumulative normal probabilities is actually very simple. For example, suppose that $X \sim N(100, 4)$ and we wish to find the probability that X is less than or equal to 104; that is, $P(X \leq 104) = F(104)$. Since the standard normal random variable is

$$Z = \frac{X - \mu}{\sigma},$$

we can *standardize* the point of interest $x = 104$ to obtain

$$z = \frac{x - \mu}{\sigma} = \frac{104 - 100}{2} = 2.$$

Now the probability that the *standard* normal random variable Z is less than or equal to 2 is equal to the probability that the *original* normal random variable X is less than or equal to 104. Expressed mathematically,

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right) = \Phi(z)$$

or

$$F(104) = \Phi(2).$$

standard normal pdf

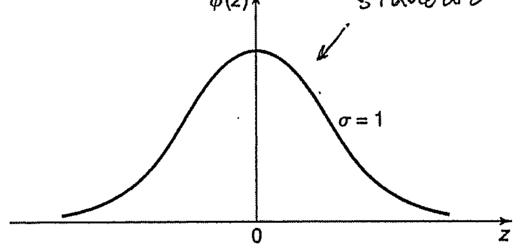

Figure 7-2 The standard normal distribution.

Appendix Table II contains cumulative standard normal probabilities for various values of z. From this table, we can read

$$\Phi(2) = 0.9772.$$

Note that in the relationship $z = (x - \mu)/\sigma$, the variable z measures the departure of x from the mean μ in standard deviation (σ) units. For instance, in the case just considered, $F(104) = \Phi(2)$, which indicates that 104 is *two* standard deviations ($\sigma = 2$) above the mean. In general, $x = \mu + \sigma z$. In solving problems, we sometimes need to use the symmetry property of φ in addition to the tables. It is helpful to make a sketch if there is any confusion in determining exactly which probabilities are required, since the area under the curve and over the interval of interest is the probability that the random variable will lie on the interval.

Example 7-1

The breaking strength (in newtons) of a synthetic fabric is denoted X, and it is distributed as $N(800, 144)$. The purchaser of the fabric requires the fabric to have a strength of at least 772 nt. A fabric sample is randomly selected and tested. To find $P(X \geq 772)$, we first calculate

$$\begin{aligned} P(X < 772) &= P\left(\frac{X - \mu}{\sigma} < \frac{772 - 800}{12}\right) \\ &= P(Z < -2.33) \\ &= \Phi(-2.33) = 0.01. \end{aligned}$$

Hence the desired probability, $P(X \geq 772)$, equals 0.99. Figure 7-3 shows the calculated probability relative to both X and Z. We have chosen to work with the random variable Z because its distribution function is tabulated.

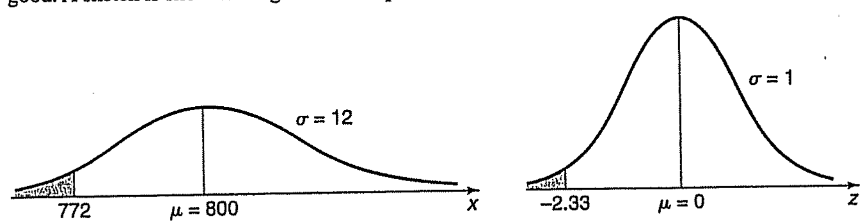

Figure 7-3 $P(X < 772)$, where $X \sim N(800, 144)$.

Example 7-2

The time required to repair an automatic loading machine in a complex food-packaging operation of a production process is X minutes. Studies have shown that the approximation $X \sim N(120, 16)$ is quite good. A sketch is shown in Fig. 7-4. If the process is down for more than 125 minutes, all equipment

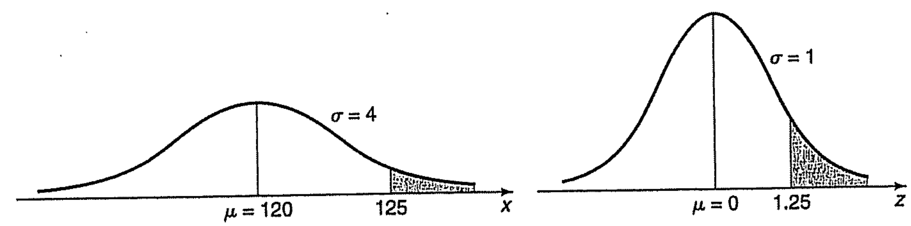

Figure 7-4 $P(X > 125)$, where $X \sim N(120, 16)$.

must be cleaned, with the loss of all product in process. The total cost of product loss and cleaning associated with the long downtime is \$10,000. In order to determine the probability of this occurring, we proceed as follows:

$$\begin{aligned} P(X>125) &= P\left(Z>\frac{125-120}{4}\right)=P(Z>1.25) \\ &= 1-\Phi(1.25) \\ &= 1-0.8944 \\ &= 0.1056. \end{aligned}$$

Thus, given a breakdown of the packaging machine, the expected cost is $E(C)=0.1056(10{,}000+C_{R_1})+0.8944(C_{R_1})$, where C is the total cost and C_{R_1} is the repair cost. Simplified, $E(C)=C_{R_1}+1056$. Suppose the management can reduce the mean of the service time distribution to 115 minutes by adding more maintenance personnel. The new cost for repair will be $C_{R_2}>C_{R_1}$; however,

$$\begin{aligned} P(X>125) &= P\left(Z>\frac{125-115}{4}\right)=P(Z>2.5) \\ &= 1-\Phi(2.5) \\ &= 1-0.9938 \\ &= 0.0062, \end{aligned}$$

so that the new expected cost would be $C_{R_2}+62$, and one would logically make the decision to add to the maintenance crew if

$$C_{R_2}+62<C_{R_1}+1056$$

or

$$C_{R_2}-C_{R_1}<\$994.$$

It is assumed that the frequency of breakdowns remains unchanged.

Example 7-3

The pitch diameter of the thread on a fitting is normally distributed with a mean of 0.4008 cm and a standard deviation of 0.0004 cm. The design specifications are 0.4000 ± 0.0010 cm. This is illustrated in Fig. 7-5. Notice that the process is operating with the mean not equal to the nominal specifications. We desire to determine what fraction of product is within tolerance. Using the approach employed previously,

$$\begin{aligned} P(0.399\le X\le 0.401) &= P\left(\frac{0.3990-0.4008}{0.0004}\le Z\le\frac{0.4010-0.4008}{0.0004}\right) \\ &= P(-4.5\le Z\le 0.5) \\ &= \Phi(0.5)-\Phi(-4.5) \\ &= 0.6915-0.0000 \\ &= 0.6915. \end{aligned}$$

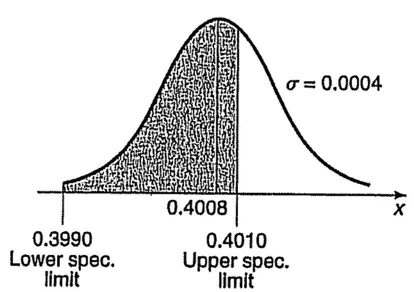

Figure 7-5 Distribution of thread pitch diameters.

As process engineers study the results of such calculations, they decide to replace a worn cutting tool and adjust the machine producing the fittings so that the new mean falls directly at the nominal value of 0.4000. Then,

$$\begin{aligned} P(0.3990\le X\le 0.4010) &= P\left(\frac{0.3990-0.4}{0.0004}\le Z\le\frac{0.4010-0.4}{0.0004}\right) \\ &= P(-2.5\le Z\le +2.5) \\ &= \Phi(2.5)-\Phi(-2.5) \\ &= 0.9938-0.0062 \\ &= 0.9876. \end{aligned}$$

We see that with the adjustments, 98.76% of the fittings will be within tolerance. The distribution of adjusted machine pitch diameters is shown in Fig. 7-6.

The previous example illustrates a concept important in quality engineering. Operating a process at the nominal level is generally superior to operating the process at some other level, if there are two-sided specification limits.

Example 7-4

This is an example of using the "Inverse" CDF

Another type of problem involving the use of tables of the normal distribution sometimes arises. Suppose, for example, that $X \sim N(50, 4)$. Furthermore, suppose we want to determine a value of X, say x, such that $P(X>x)=0.025$. Then,

$$P(X>x)=P\left(Z>\frac{x-50}{2}\right)=0.025$$

This is sort of like trig problems
$\cos(x) = 2$
$x = \cos^{-1}(2)$

or

$$P\left(Z\le\frac{x-50}{2}\right)=0.975,$$

so that, reading the normal table "backward," we obtain

$$\frac{x-50}{2}=1.96=\Phi^{-1}(0.975)$$

and thus

$$x=50+2(1.96)=53.92.$$

There are several symmetric intervals that arise frequently. Their probabilities are

$$\begin{aligned} P(\mu-1.00\sigma\le X\le\mu+1.00\sigma) &= 0.6826, \\ P(\mu-1.645\sigma\le X\le\mu+1.645\sigma) &= 0.90, \\ P(\mu-1.96\sigma\le X\le\mu+1.96\sigma) &= 0.95, \\ P(\mu-2.57\sigma\le X\le\mu+2.57\sigma) &= 0.99, \\ P(\mu-3.00\sigma\le X\le\mu+3.00\sigma) &= 0.9978. \end{aligned} \qquad (7\text{-}9)$$

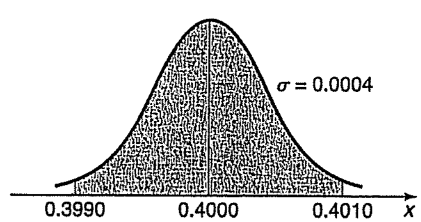

Figure 7-6 Distribution of adjusted machine pitch diameters.