EECS 360 Short Quiz #4

Signal and System Analysis

April 5, 2012

Name: KEY

Open book, open notes, no calculator. Be neat, write legibly. For full credit you must show all work and justify each answer. You may write on the both sides of the paper, and use additional sheets of paper if needed.

1. (25 %) Prove the time scaling property of the continuous-time Fourier transform (CTFT), which is

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

assume $a>0$

$$\mathcal{F}(x(at)) = \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t}\, dt$$

← change of variable, Let $\lambda = at$, $\Rightarrow t = \lambda/a$ $\Rightarrow dt = \frac{d\lambda}{a}$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda/a}\, d\lambda$$

$$= \frac{1}{a} X\left(\frac{f}{a}\right)$$

Again assume $a>0$

$$\mathcal{F}(x(-at)) = \int_{-\infty}^{\infty} x(-at) e^{-j2\pi f t}\, dt$$

← Let $\lambda = -at$ $\Rightarrow t = -\lambda/a$ $\Rightarrow dt = -d\lambda/a$

$$= \frac{1}{-a}\int_{\infty}^{-\infty} x(\lambda) e^{-j2\pi\left(-\frac{f}{a}\right)\lambda}\, d\lambda$$

exchange limits of integration, cancels minus sign

$$= \frac{1}{a} X\left(-\frac{f}{a}\right)$$

Relax assumption about $a>0$, either way the term out front is positive

$$\boxed{x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)}$$

2. (50 %) Consider an LTI system defined by the difference equation:

$$y[n] = -2x[n] + 4x[n-1] - 2x[n-2]$$

(a) Determine the impulse response of the system, $h[n]$.

(b) Determine the frequency response of the system, $H(F)$ [you can also work this problem using the $H(e^{j\Omega})$ notation if you prefer]. Express your answer in the form

$$H(F) = A(F)e^{-j2\pi F n_d}$$

where $A(F)$ is a real function of F. Explicitly specify $A(F)$ and the delay n_d of this system.

(c) Sketch a plot of the magnitude $|H(F)|$ and a plot of the phase $\angle H(F)$.

(d) Suppose that the input to the system is

$$x_1[n] = 1 + e^{j\pi n/2}, \qquad -\infty < n < +\infty.$$

Use the frequency response function to determine the corresponding output $y_1[n]$.

(a) Time-domain method: let $x[n] = \delta[n]$, then $y[n]$ must be $h[n]$

$$\boxed{h[n] = -2\delta[n] + 4\delta[n-1] - 2\delta[n-2]}$$

Frequency-domain method:

$$Y(F) = -2X(F) + 4e^{-j2\pi F}X(F) - 2e^{-j4\pi F}X(F)$$

$$\Rightarrow \frac{Y(F)}{X(F)} = \boxed{H(F) = -2 + 4e^{-j2\pi F} - 2e^{-j4\pi F}}$$

$$\downarrow \mathcal{F}^{-1}$$

$$\boxed{h[n] = -2\delta[n] + 4\delta[n-1] - 2\delta[n-2]}$$

start with

(b) Factor out $e^{-j2\pi F}$

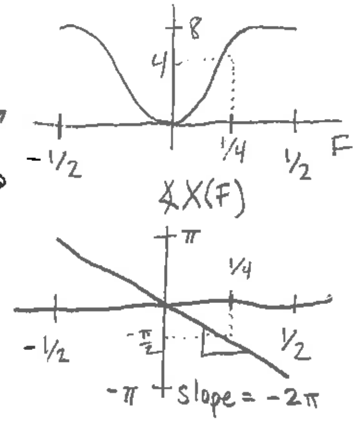
$$H(F) = e^{-j2\pi F}\left[4 - 2e^{j2\pi F} - 2e^{-j2\pi F}\right]$$

$$= \underbrace{[4 - 4\cos(2\pi F)]}_{A(F)} e^{-j2\pi F}$$

$\leftarrow \boxed{n_d = 1}$

$\leftarrow \boxed{A(F) = 4 - 4\cos(2\pi F)}$

(c)



(d) $X_1(F) = \underbrace{\delta_1(F)}_{} + \underbrace{\delta_1(F - 1/4)}_{} \longrightarrow Y_1(F) = H(F)X_1(F)$

we need to evaluate $H(F)$ at $F = 0$

$H(F)|_{F=0} = 0$

This system rejects DC

we need to evaluate $H(F)$ at $F = 1/4$

$H(F)|_{F=1/4} = -4j$

$$\boxed{= -4j\,\delta_1(F - 1/4)}$$

$$\downarrow \mathcal{F}^{-1}$$

$$y_1[n] = -4j\,e^{j\pi n/2}$$

3. (25 %) A continuous-time signal $x(t)$ has the Fourier transform

$$X(f) = \frac{1}{j2\pi f + b}$$

where b is a constant. Determine the Fourier transform $G(f)$ of the following signals:

(a) $g(t) = x(5t - 4)$

(b) $g(t) = tx(t)$

(c) $g(t) = x(t)e^{j2t}$

(d) $g(t) = \dfrac{d^2x(t)}{dt^2}$

(a) Time scaling, time shifting

$$x\left(5\left(t - \tfrac{4}{5}\right)\right) \xleftrightarrow{\mathcal{F}} \frac{1}{5} X\left(\frac{f}{5}\right) e^{-j2\pi f \cdot 4/5}$$

$$= \frac{e^{-j\frac{8\pi}{5}f}}{5} \cdot \frac{1}{j\frac{2\pi}{5}f + b}$$

(b) $$t\,x(t) \xleftrightarrow{\mathcal{F}} \frac{-j}{2\pi} \frac{d}{df} X(f)$$

$$= \frac{-j}{2\pi} \frac{d}{df} (j2\pi f + b)^{-1}$$

$$= \frac{-j}{2\pi} (-1)(j2\pi f + b)^{-2}(j2\pi) = \frac{-1}{(j2\pi f + b)^2}$$

(c) $$x(t)\,e^{j2t} \longleftrightarrow X\left(f - \tfrac{1}{\pi}\right)$$

$$= \frac{1}{j2\pi f - j2 + b}$$

(d) $$\frac{d^2x(t)}{dt^2} = -(2\pi f)^2 X(f)$$

$$= \frac{-(2\pi f)^2}{j2\pi f + b}$$