EECS 360 Short Quiz #4

Signal and System Analysis

November 15, 2012

Name: KEY

Open book, open notes, no calculator. Be neat, write legibly. For full credit you must show all work and justify each answer. You may write on the both sides of the paper, and use additional sheets of paper if needed.

1. (25 %) Derive the convolution property of the Fourier transform

$$x[n] * h[n] \overset{\mathcal{F}}{\longleftrightarrow} X(F)H(F).$$

Be sure to show all your work!

[If you don't know how to begin, write down what $X(F)$ is equal to, then write down what $H(F)$ is equal to, then write down what convolution is equal to, and then take the Fourier transform of convolution.]

$$X(F) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi F n} \qquad H(F) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi F n}$$

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m]$$

$$Y(F) = \sum_{n=-\infty}^{\infty} y[n] e^{-j2\pi F n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x[m] h[n-m] \right] e^{-j2\pi F n}$$

change of variables, let $\ell = n-m \Rightarrow n = \ell + m$

$$= \sum_{\ell=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] h[\ell] e^{-j2\pi F(\ell+m)}$$

$$= \underbrace{\sum_{\ell=-\infty}^{\infty} h[\ell] e^{-j2\pi F \ell}}_{H(F)} \cdot \underbrace{\sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi F m}}_{X(F)}$$

2. (25 %) Find the inverse DTFT of

$$X(e^{j\Omega}) = \frac{1}{1 - 0.3e^{-j(\Omega - \frac{\pi}{3})}}.$$

$$X(e^{j\Omega}) = G(e^{j(\Omega - \frac{\pi}{3})})$$

where $G(e^{j\Omega}) = \dfrac{1}{1 - 0.3e^{-j\Omega}}$

Table of DTFT pairs

$$g[n] = (0.3)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} G(e^{j\Omega}) = \frac{1}{1 - 0.3e^{-j\Omega}}$$

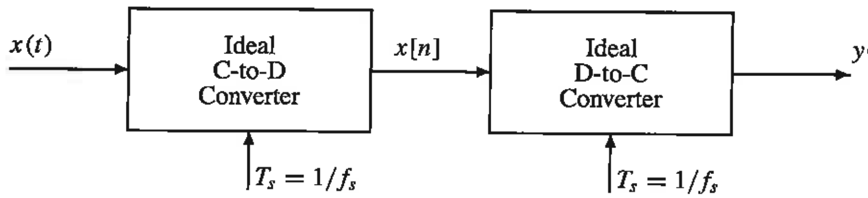
Freq shift Prop

$$g[n] e^{j\frac{\pi}{3}n} \longleftrightarrow G(e^{j(\Omega - \frac{\pi}{3})})$$

$$\Rightarrow x[n] = e^{j\frac{\pi}{3}n} (0.3)^n u[n]$$

3. (50 %) The figure below shows a simple sampling and reconstruction system. Assume that the sampling rates of the continuous-to-discrete (C-to-D) and discrete-to-continuous (D-to-C) converters are equal. The input to the C-to-D converter is

$$x(t) = 2\cos(100\pi t + \pi/2) + \cos(300\pi t).$$



(a) If the output of the D-to-C converter is equal to the input $x(t)$, i.e.,

$$y(t) = 2\cos(100\pi t + \pi/2) + \cos(300\pi t),$$

what general statement can you make about the sampling frequency f_s in this case?

(b) Now, suppose the sampling rate is known to be $f_s = 250$ samples/second. Determine the discrete-time signal $x[n]$ in this instance; express your answer in the same form as $x(t)$ above, i.e., as a sum of cosines. [Make sure that all frequencies in your answer are positive and less than π radians/sample (or less than 1/2 cycles/sample).]

(c) Plot the spectrum of the signal in part (b) over the range of frequencies $-\pi \le \Omega < \pi$ (or $-1/2 \le F < 1/2$). Carefully label your plot with the frequency, amplitude, and phase of each spectral component.

(d) Now, suppose the output of the D-to-C converter is

$$y(t) = 2\cos(100\pi t + \pi/2) + 1.$$

Determine the value of the sampling frequency f_s in this case. [The input $x(t)$ is still the same as above.]

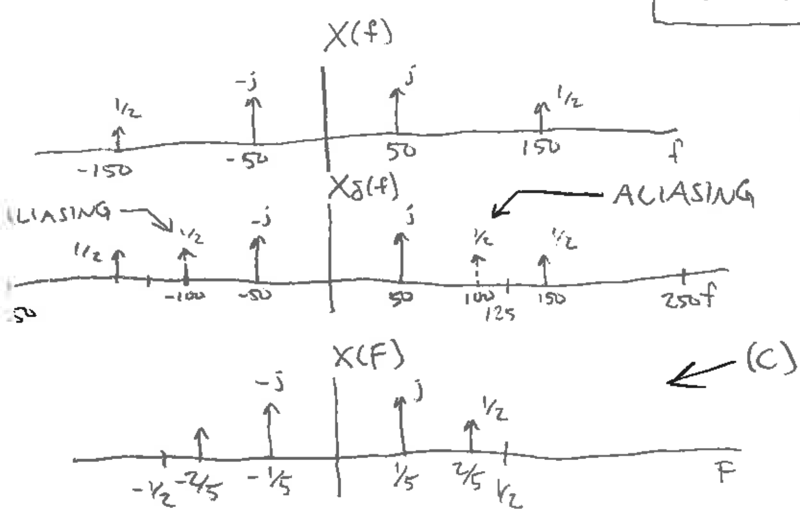
(a) $x(t)$ is a band limited signal with $f_m = 150$ Hz $[\omega_m = 300\pi$ rad/s$]$

If $y(t) = x(t)$ then it is safe to say that $f_s > 2 \times 150$ Hz, or $\boxed{f_s > 300\text{Hz}}$

(b) $x[n] = x(nT_s) = x(n/250) = 2\cos(2\pi(\frac{1}{5})n + \pi/2) + \cos(2\pi(\frac{3}{5})n)$ subtract $2\pi n$

$= 2\cos(2\pi(\frac{1}{5})n + \pi/2) + \cos(2\pi(-\frac{2}{5})n)$

$\boxed{= 2\cos(2\pi(\frac{1}{5})n + \pi/2) + \cos(2\pi(\frac{2}{5})n)}$



(d) The 2nd term aliases to DC

We know that the frequency f_s aliases to DC, therefore, because the frequency of the 2nd term is 150

$\Rightarrow f_s = 150$

$\cos(300\pi t)\big|_{t = 1/150} = \cos(2\pi(\frac{150}{150})n)$

$= 1$