Inverse of an "echo" channel, or a "multipath" communications channel.

Consider a system with input/output relationship

$$y[n] = x[n] + a\, x[n-1]$$

Find a causal inverse system that recovers $x[n]$ from $y[n]$.

Solution

The impulse response of this system is:

set $x[n] = \delta[n]$
rename $y[n]$ as $h[n]$

$$h[n] = \delta[n] + a\delta[n-1]$$

The inverse system $h_I[n]$ satisfies the condition

$$h[n] * h_I[n] = \delta[n] \qquad (1)$$

Because $h_I[n]$ is required to be causal, we have $h_I[n] = 0$ for $n < 0$

Equation 1 tells us $h[n] * h_I[n] = \delta[n]$

$$\Rightarrow (\delta[n] + a\delta[n-1]) * h_I[n] = \delta[n]$$

$$\Rightarrow h_I[n] + a h[n-1] = \delta[n]$$

when $n=0$ we have $h_I[0] + a \cdot \emptyset = 1 \Rightarrow h[0] = 1$

when $n=1$ we have $h_I[1] + a h[0] = 0 \Rightarrow h[1] = -a$

when $n=2$ we have $h_I[2] + a h[1] = 0 \Rightarrow h[2] = +a^2$

$$\Rightarrow \text{by induction we have} \quad \boxed{h_I[n] = (-a)^n u[n]}$$