1. Let $X$ be a random variable with mean $\mu_x$ and variance $\sigma^2_x$. Show that the random variable $Y = X + b$ has mean $\mu_y = \mu_x + b$ and variance $\sigma^2_y = \sigma^2_x$. This problem demonstrates that any random variable can be thought of as a zero mean random variable plus a constant; the addition of the constant changes the mean but not the variance.

2. Let $X$ be a random variable with mean $\mu_x$ and variance $\sigma^2_x$. Show that the random variable $Y = aX$ has mean $\mu_y = a\mu_x$ and variance $\sigma^2_y = a^2\sigma^2_x$.

3. Let $X$ be a random variable with mean $\mu_x$ and variance $\sigma^2_x$. Suppose we want to obtain a version of $X$ that has zero mean and unit variance. In other words, we want to obtain the random variable $Z = h(X)$ such that $Z$ has zero mean and unit variance. Based on the results from the first two problems, show the exact relationship between $X$ and $Z$.

4. Let $W$ and $X$ be random variables with respective means $\mu_w$ and $\mu_x$ and variances $\sigma^2_w$ and $\sigma^2_x$. Show that the random variable $Y = W + X$ has mean $\mu_y = \mu_w + \mu_x$.

5. Let $W$ and $X$ be independent random variables with zero mean and respective variances $\sigma^2_w$ and $\sigma^2_x$. Show that the random variable $Y = W + X$ has variance $\sigma^2_y = \sigma^2_w + \sigma^2_x$. It is reasonable to ask: What if $W$ and $X$ have non-zero means? Does $\sigma^2_y$ change? Based on the result of Problem 1, argue that $\sigma^2_y$ does not change when we add a non-zero mean to $W$ and $X$.

6. Let $\{X_i\}$ be a set of $n$ independent random variables, each with an individual mean $\mu$ and variance $\sigma^2$. Based on all of the above, show that the sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

has mean $\mu_{\bar{X}} = \mu$ and variance $\sigma^2_{\bar{X}} = \sigma^2/n$. 
