Symmetric Information Rate
EECS 769: Information Theory
Fall 2018

In this project you will perform the numerical computation of a quantity called the *symmetric information rate* (SIR), which we will use as a proxy for the channel capacity.

Background Material

We will consider the Gaussian channel

$$Y = X + Z \tag{1}$$

where $Z \sim \mathcal{N}(0, N)$. We assume that X is drawn from a discrete alphabet and we further assume that X has the uniform distribution. As such, we can compute the mutual information between X and Y, $I(X;Y)$. However, unlike the case of channel capacity, we do not have the task of maximizing $I(X;Y)$ with respect to the input distribution, because it is already fixed as the uniform distribution. The value of $I(X;Y)$ we obtain with this scenario we call the *symmetric information rate* (SIR).

In order to compute the SIR we need the conditional entropy

$$H(X|Y) = \int f(y) H(X|Y=y)\, dy \tag{2}$$

$$= \mathrm{E}_y\left[H(X|Y=y)\right] \tag{3}$$

where

$$H(X|Y=y) = -\sum_{x \in \mathcal{X}} p(X=x|Y=y) \log\left[p(X=x|Y=y)\right] \tag{4}$$

and

$$p(X=x|Y=y) = Ae^{-(y-x)^2/2N} \quad \text{for all } x \in \mathcal{X} \tag{5}$$

with the value of the constant A being chosen so that $p(X=x|Y=y)$ sums to unity.

Because of the difficulties of integrating with respect to the pdf of Y, we will apply the law of large numbers so that

$$H(X|Y) = \mathrm{E}_y\left[H(X|Y=y)\right] = \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} H(X|Y=y_k) \tag{6}$$

where y_k is a sample realization of (1) and there are K such realizations in total.

The SIR can thus be obtained by Monte Carlo simulation as

$$I(X;Y) = H(X) - H(X|Y) \tag{7}$$

$$= \log|\mathcal{X}| - \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} H(X|Y=y_k) \tag{8}$$

Procedure for Evaluating the SIR

Let EsNoDB be a set of desired values of the symbol-based signal-to-noise ratio (SNR), E_s/N_0, and let $\mathrm{EsNoDB}(l)$ be the l-th value of this set. For example, $\mathrm{EsNoDB} = \{-40, -39, \ldots + 39, +40\}$. Let $\mathbf{C}$ be a M-ary quadrature amplitude modulation (MQAM) constellation (i.e., a look-up table, or a column vector) with elements $C(i)$, where $i \in \{0, 1, \ldots, M-1\}$. The constellation is normalized so that its average symbol energy is unity, i.e. $E_s = 1$. Because of this normalization, when $\mathrm{EsNoDB}(l)$ is specified then the correspondingvalue of N_0 is $N_0(l) = 10^{-\mathrm{EsNoDB}(l)/10}$. For every $\mathrm{EsNoDB}(l)$, perform the following steps:

1. Generate a random index $i_k \in \{0, 1, \ldots, M-1\}$ according to the uniform distribution.

2. Generate $z_k = z_{k,R} + jz_{k,I}$, where $z_{k,R}$ and $z_{k,I}$ are independent Gaussian random variables each with zero mean and variance $N_0(l)/2$.

3. Obtain $y_k = C(i_k) + z_k$.

4. Compute the conditional pmf $p(X = C(i)|y_k)$, which has M entries, one for each $C(i)$. This can be done in "vector" fashion as

$$p(\mathbf{C}|y_k) = Ae^{-|y_k - \mathbf{C}|^2/N_0(l)} \tag{9}$$

where $p(\mathbf{C}|y_k)$ is a column vector whose entries must first be computed with $A = 1$ and then normalized so they sum to unity.

5. Compute $H(X|Y = y_k)$ as

$$H(X|Y = y_k) = -\sum_{i=0}^{M-1} p(C(i)|y_k) \log\left[p(C(i)|y_k)\right] \tag{10}$$

Recall that we use the convention $0 \log 0 = 0$; however, the computer might return the value $0 \log 0 =$ NaN (not a number), which should be replaced by zero.

6. Repeat Steps 1–5 for K trials, $k \in \{0, 1, \ldots, K-1\}$, where K is a large number.

7. Compute the SIR for the current value of EsNoDB(l) as

$$I(X;Y) = \log M - \frac{1}{K}\sum_{k=0}^{K-1} H(X|Y = y_k) \tag{11}$$

Interpretation of the Results

The SIR, like the channel capacity, indicates an information rate, which we will call R. A channel encoding scheme takes information bits and adds additional "redundant" (or "parity") bits so that channel errors can be corrected and error-free communication can take place. The encoded bits (information plus parity) are grouped into m-tuples and converted to M-ary symbols, where $m = \log_2 M$. Because we are transmitting one symbol per channel use, the units of R are information bits per channel symbol. This gives us an identity that relates the symbol energy and the information bit energy:

$$E_s = RE_b \qquad\qquad E_b = \frac{E_s}{R} \tag{12}$$

With the procedure outlined above, we have tabulated a set of rates for a set of EsNoDB for a given MQAM constellation $\mathbf{C}$. Using (12), we can obtain E_b by dividing each E_s by its respective R. In dB, this is done with EbNoDB(l) = EsNoDB$(l) - 10\log_{10}\left[I(X;Y)\right]$. This yields a new x-axis for the table in terms of E_b/N_0.

Exercise

1. Compute the SIR vs. E_b/N_0 for BPSK.

2. Compute the SIR vs. E_b/N_0 for QPSK.

3. Compute the SIR vs. E_b/N_0 for 8PSK.

4. Compute the SIR vs. E_b/N_0 for 16-QAM.

5. Compute the SIR vs. E_b/N_0 for 64-QAM.

6. Plot these curves on the same axis with the capacity of the bandwidth-constrained AWGN channel, $\frac{C}{W} = \log\left[1 + \frac{C}{W}\frac{E_b}{N_0}\right]$. E_b/N_0 should be in dB (i.e. EbNoDB). You might consider making two plots, one where the y-axis is in the linear scale and another where the y-axis is in the log scale.