Discrete Time Convolution

Lab 4

Look at these two signals

$$x[n] = 1, \qquad 0 \le n \le 4$$

$$\mathrm{h}[n] = 1, \qquad -2 \le n \le 2$$

m	x[m]	h[m]
-6	0	0
-5	0	0
-4	0	0
-3	0	0
-2	0	1
-1	0	1
0	1	1
1	1	1
2	1	1
3	1	0
4	1	0
5	0	0
6	0	0

Suppose we wanted their discrete time convolution:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

This infinite sum says that a single value of y, call it $y[n]$ may be found by performing the sum of all the multiplications of $x[m]$ and $h[n-m]$ at every value of m.

Direct Approach using Convolution Sum

Example:

$$y[0] = \sum_{m=-\infty}^{\infty} x[m]h[0-m]$$

$$\Rightarrow y[0] = \cdots + x[-2]h[0-(-2)] + x[-1]h[0-(-1)]$$
$$+x[0]h[0-(0)] + x[1]h[0-(1)]$$
$$+x[2]h[0-(2)] + x[3]h[0-(3)]$$
$$+x[4]h[0-(4)] + \cdots$$

$$\Rightarrow y[0] = \cdots + x[-2]h[2] + x[-1]h[1]$$
$$+x[0]h[0] + x[1]h[-1] + x[2]h[-2]$$
$$+x[3]h[-3] + x[4]h[-4] + \cdots$$

$$\Rightarrow y[0] = \cdots + (0)(1) + (0)(1)$$
$$+(1)(1) + (1)(1) + (1)(1)$$
$$+(1)(0) + (1)(0) + \cdots$$

$$\therefore y[0] = 3$$

m	x[m]	h[m]
-6	0	0
-5	0	0
-4	0	0
-3	0	0
-2	0	1
-1	0	1
0	1	1
1	1	1
2	1	1
3	1	0
4	1	0
5	0	0
6	0	0

Table Method

Table Method

m	x[m]	h[m]	h[0-m]	x[m]h[-m]
-6	0	0	h[-(-6)]=h[6]=0	0
-5	0	0	h[5]=0	0
-4	0	0	0	0
-3	0	0	0	0
-2	0	1	1	0
-1	0	1	1	0
0	1	1	1	1
1	1	1	1	1
2	1	1	1	1
3	1	0	0	0
4	1	0	0	0
5	0	0	0	0
6	0	0	h[-6]=0	0

The sum of the last column is equivalent to the convolution sum at y[0]!

$$\therefore y[0] = 3$$

Consulting a larger table gives more values of y[n]

m	x[m]	h[0-m]	h[-1-m]	h[-2-m]	h[-3-m]
-6	0	h[-(-6)]=h[6]=0	h[-1-(-6)]=h[5]=0	0	0
-5	0	h[5]=0	0	0	1
-4	0	0	0	1	1
-3	0	0	1	1	1
-2	0	1	1	1	1
-1	0	1	1	1	1
0	1	1	1	1	0
1	1	1	1	0	0
2	1	1	0	0	0
3	1	0	0	0	0
4	1	0	0	0	0
5	0	0	0	0	0
6	0	h[-6]=0	0	0	0

Notice what happens as decrease n, h[n-m] shifts up in the table (moving forward in time).

$$\therefore y[-3] = 0$$

$$\therefore y[-2] = 1$$

$$\therefore y[-1] = 2$$

$$\therefore y[0] = 3$$

$y[n]$ for $n > 0$

m	x[m]	h[1-m]	h[2-m]	h[3-m]	h[4-m]	h[5-m]	h[6-m]	h[7-m]
-6	0	0	0	0	0	0	0	0
-5	0	0	0	0	0	0	0	0
-4	0	0	0	0	0	0	0	0
-3	0	0	0	0	0	0	0	0
-2	0	0	0	0	0	0	0	0
-1	0	1	0	0	0	0	0	0
0	1	1	1	0	0	0	0	0
1	1	1	1	1	0	0	0	0
2	1	1	1	1	1	0	0	0
3	1	1	1	1	1	1	0	0
4	1	0	1	1	1	1	1	0
5	0	0	0	1	1	1	1	1
6	0	0	0	0	1	1	1	1

$\therefore y[1] = 4$

$\therefore y[2] = 5$

$\therefore y[3] = 4$

$\therefore y[4] = 3$

$\therefore y[5] = 2$

$\therefore y[6] = 1$

$\therefore y[7] = 0$

Alternative Analytical Approach

(sometimes called the Echo method)

If $x[n]$ or $h[n]$ is pretty short we can represent $x[n]$ or $h[n]$ with a linear combination of discrete impulses:

$$x[n] = 1, \qquad 0 \leq n \leq 4$$

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

$$\mathrm{h}[n] = 1, \qquad -2 \leq n \leq 2$$

$$\mathrm{h}[n] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n+1] + \delta[n-2]$$

$$y[n] = x[n] * h[n]$$

$$But\ h[n] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n+1] + \delta[n-2]$$

$$\Rightarrow y[n] = x[n] * \{\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n+1] + \delta[n-2]\}$$

$Using\ the\ distributive\ property,$

$$\Rightarrow y[n] = x[n] * \delta[n+2] + x[n] * \delta[n+1] + x[n] * \delta[n] + x[n] * \delta[n+1] + x[n] * \delta[n-2]$$

$Using\ the\ sampling\ property,$

$$\Rightarrow y[n] = x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2]$$

$So\ the\ output\ is\ just\ a\ linear\ combintation\ of\ shifted\ input\ signals!$

$$\Rightarrow y[n] = x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2]$$

$$But\ x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

$$\Rightarrow y[n] = \{\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]\}$$

$$+\{\delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]\}$$

$$+\{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]\}$$

$$+\{\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]\}$$

$$+\{\delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + \delta[n-6]\}$$

So many terms! ...But there are many “copies” of several terms.

Combining like terms:

$$\Rightarrow y[n] = \{\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]\}$$

$$+\{\delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]\}$$

$$+\{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]\}$$

$$+\{\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]\}$$

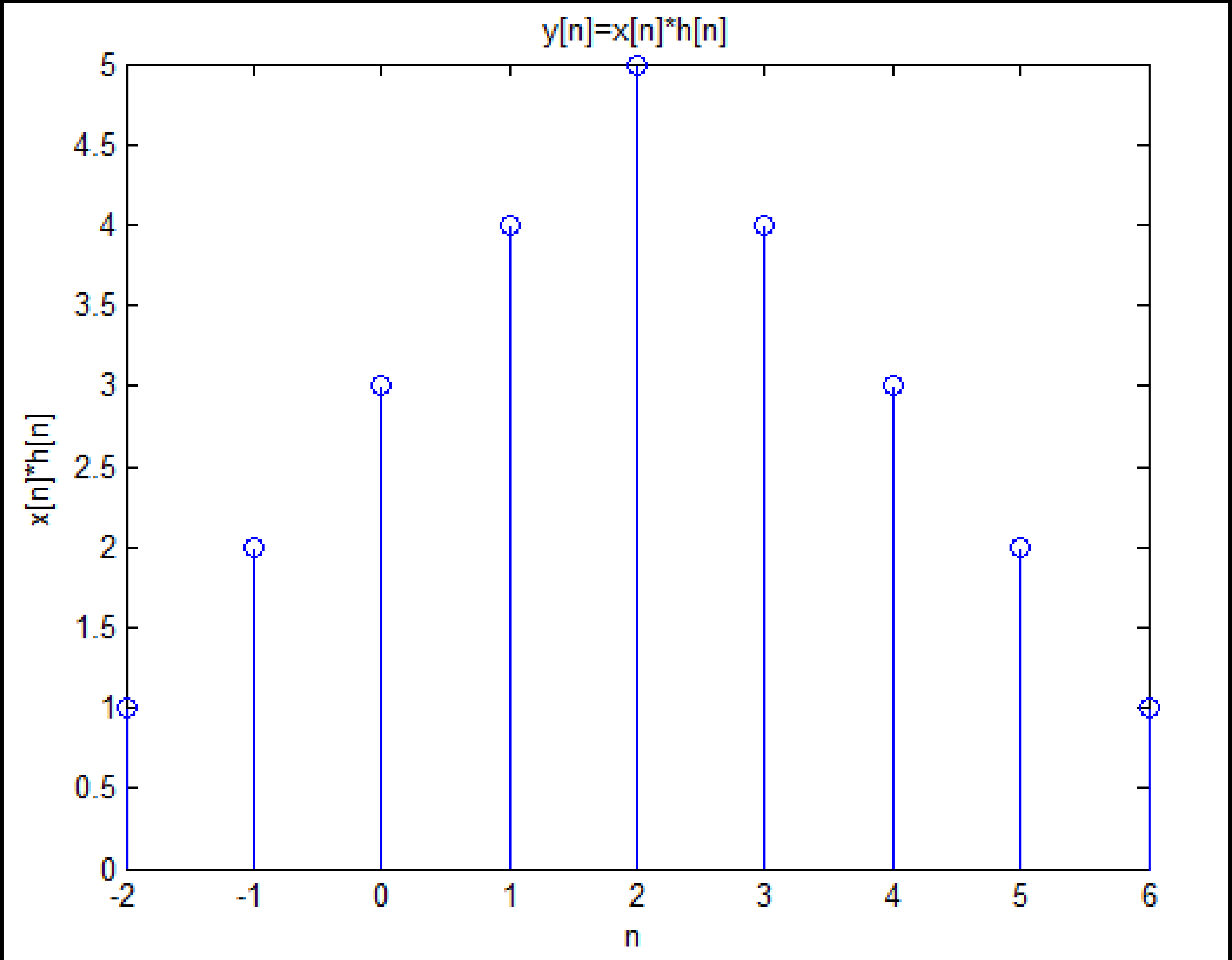
$$+\{\delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + \delta[n-6]\}$$

$$\therefore y[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 4\delta[n-1] + 5\delta[n-2]$$

$$+4\delta[n-3] + 3\delta[n-4] + 2\delta[n-5] + \delta[n-6]$$

Note that this is the same result as the table method. Also $3\delta[n]$ is an alternative description for $y[0] = 3$ found using the convolution sum.

THE RESULT



$\therefore y[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 4\delta[n-1] + 5\delta[n-2] + 4\delta[n-3] + 3\delta[n-4] + 2\delta[n-5] + \delta[n-6]$

Using MATLAB

conv()

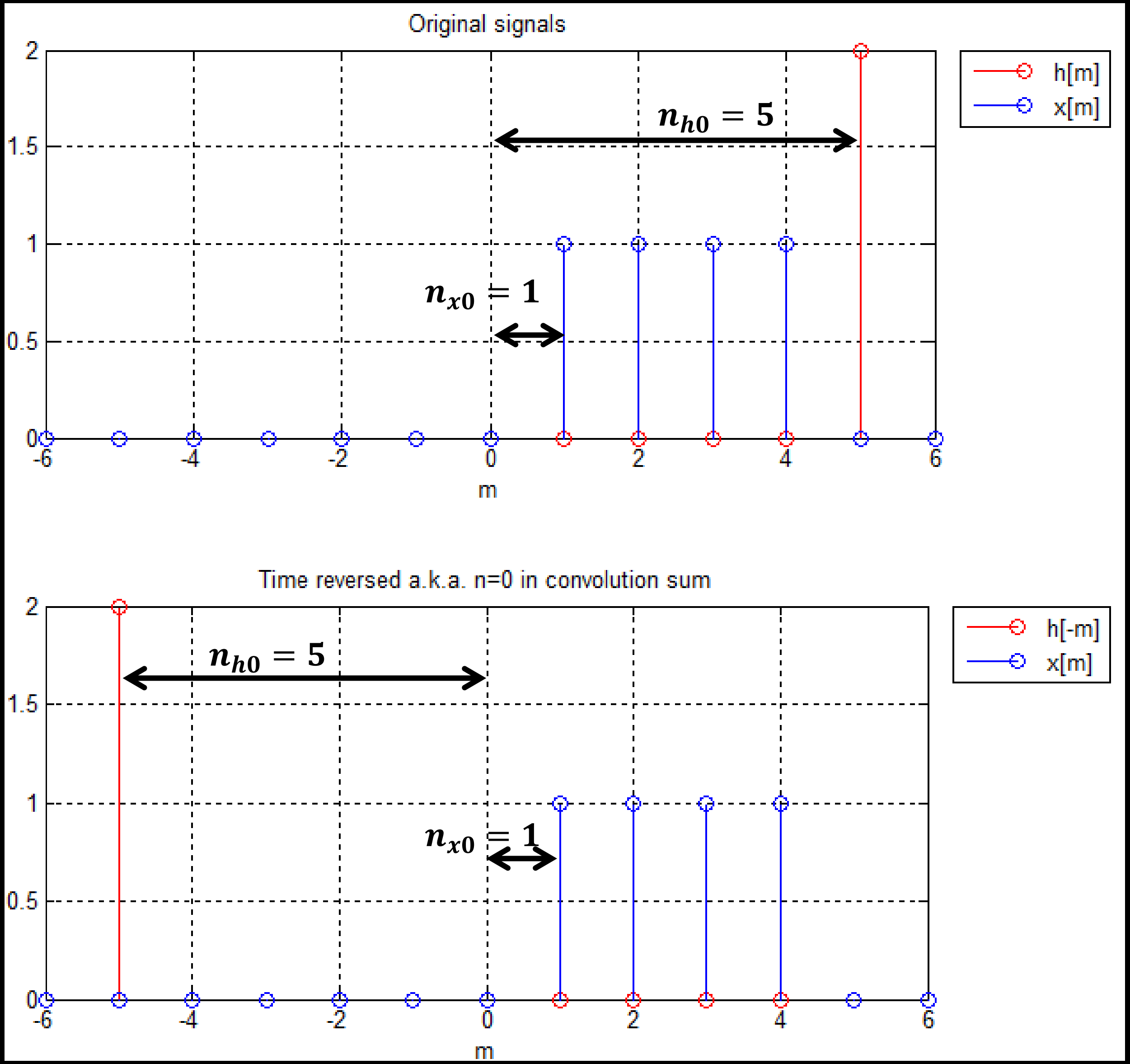
Code

```
n=-2:1:6;  %How did I know this?

y=[1 2 3 4 5 4 3 2 1];

stem(n,y)

title('y[n]=x[n]*h[n]')
ylabel('x[n]*h[n]')
xlabel('n')
```

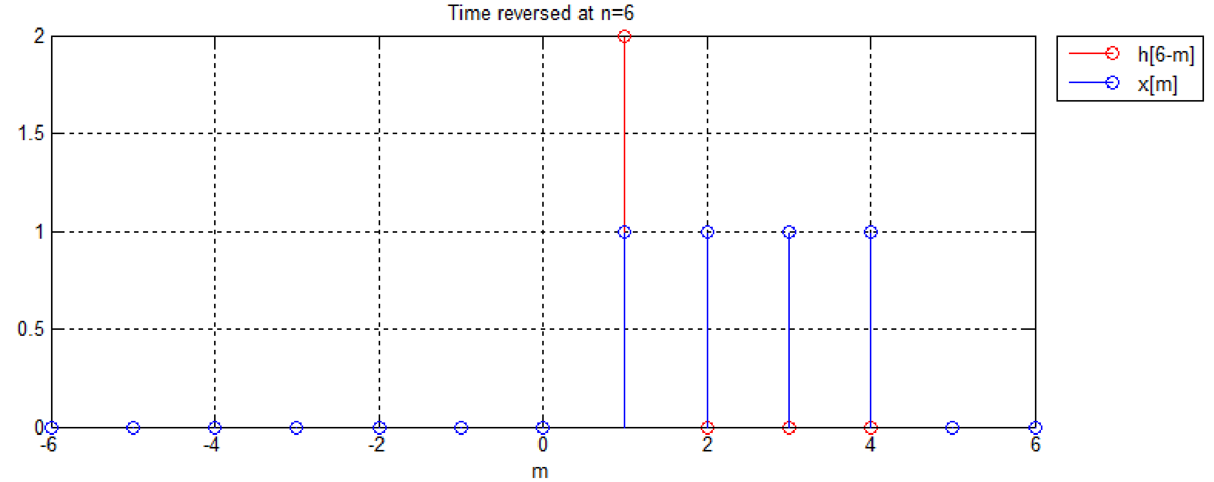
Q: How do I tell MATLAB where to plot the convolution?

A: If the time of the first element of x is n_{x0} and the time of the first element of h is n_{h0} then the time of the first element of y is n_{x0}+n_{h0}. Likewise the time of the last element of y is n_{x1}+n_{h1} where n_{x1} corresponds to the time of the last element of x and n_{h1} corresponds to the time of the last element of h.

$$\mathrm{n}_{\mathrm{x}0} + n_{h0} \leq n < n_{x1} + n_{h1}$$



Time reversed signal moves over $n_{x0} + n_{h0} = 6$ places before product is nonzero!



This is a true statement even if n_{x0} or n_{h0}, or both, are negative!

So for our problem,

$$x[n] = 1, \qquad 0 \le n \le 4$$

$$h[n] = 1, \qquad -2 \le n \le 2$$

$$\rightarrow \quad \mathrm{n}_{\mathrm{x0}} + n_{h0} \le n < n_{x1} + n_{h1}$$

$$(0) + (-2) \le n < (4) + (2)$$

$$-2 \le n < 6$$

Sources

Hsu, H. (2011). *Schaum's Outline of Signals and Systems* (2nd ed.). New York, NY: McGraw-Hill.

Roberts, M.J. (2012). *Signals and Systems: Analysis Using Transform Methods and MATLAB®* (2nd ed.). New York, NY: McGraw-Hill.