Detecting and Calculating Conic Sections in the Intersection of Two Natural Quadric Surfaces
Part I: Theoretical Analysis

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Abstract

One of the most challenging aspects of the surface-surface intersection problem is the proper disposition of degenerate configurations. The topology of the intersection as well as the basic geometric representation of the curve itself is often at stake. By Bezout's Theorem, two quadric surfaces always intersect in a degree four curve in complex projective space. This degree four curve is degenerate if it splits into two (possibly degenerate) conic sections. In theory the presence of such degeneracies can be detected using classical algebraic geometry. Unfortunately it has proven to be extremely difficult in practice to make computer implementations of such methods reliable numerically. Here we use classical algebraic methods to characterize precisely those situations in which two natural quadric surfaces (spheres, right circular cylinders, and right circular cones) intersect in planar curves. We then re-interpret the resulting algebraic conditions geometrically so that robust geometric schemes can be used in the actual computer implementation. This paper presents the background algebraic analysis and shows how to interpret the resulting conditions geometrically. It is likely to be of interest primarily to theoreticians as none of this work is implemented in computer code. A companion paper [8] discusses how the results of this paper can be used to derive geometric algorithms for detecting the presence of and computing the descriptions for these conic sections. Readers who are looking only for robust and efficient algorithms, but who do not wish to examine the derivations and proofs of correctness and completeness need only study the companion paper [8].

1.0 Introduction

The surface-surface intersection problem in geometric modeling continues to be a challenging one for system developers. Especially problematic are those situations in which the curve of intersection between two surfaces is in some fashion degenerate. This may occur when the two surfaces are tangent -- either along a curve or at a finite number of points -- or, as we shall see, in a variety of other situations, not all of which are entirely intuitive. Such problems are challenging even when the domain of surfaces is restricted to the quadrics; witness the considerable research attention this subject continues to draw [2,7,11], even though most of the basic results have been known in computer-based implementations for quite some time [3,6,12].

The importance of detecting the presence of conic sections in quadric surface intersections is well established [2,7,9,12]. Often cited advantages include: more efficient and precise data base representations, more reliable tests for common curves in the boundary evaluation algorithm of solid modeling, and more accurate analytical operations such as intersections and arc length and tangent direction computations. Another benefit has recently been discovered in the construction of blending surfaces. In general, a blend surface between two arbitrary quadric surfaces will have degree four. Warren [14,15] has shown that quadric surfaces which intersect in planar curves can be blended with surfaces of degree three.

The majority of the published results dealing with quadric surface intersections are based either on the methods of classical algebraic geometry [2,6,12] or on case-by-case geometric analysis [7,11]. Algebraic methods are valuable for insuring completeness and generality. That is, one can prove rigorously that all possible geometric configurations have been properly taken into account. It has, unfortunately, proven to be quite difficult to make algorithms based solely on algebraic geometry sufficiently reliable numerically for production use in solid modeling systems. Geometric methods, on the other hand, have proven to be faster and more reliable numerically, but they are based on a case-by-case analysis in which it is much more difficult, if not impossible, to prove that all cases are handled properly.

Our approach is to tap the best of both worlds by limiting the scope of surfaces covered to those most commonly used (namely the natural quadrics [5]), applying algebraic geometry to characterize with certainty all situations under which a pair of natural quadrics have planar intersections, and finally reinterpreting the algebraic conditions in geometrically invariant terms so that robust geometric algorithms can be implemented to detect and calculate the resulting planar intersections. This paper presents the results of the algebraic analysis and shows how to reinterpret them in geometric terms. A companion paper [8] outlines precisely how the various planar intersections can be computed.

We proceed in the following fashion. In Section 2 we survey previous work in the field and we provide most of the theory necessary for understanding our own approach. In Section 3 we give a brief overview of our method. Section 4 is the bulk of this paper. Here we provide a case-by-case analysis of the degenerate intersections of all possible pairs of natural quadric surfaces. This analysis is necessarily quite long and detailed. However the final results are easy to state. These are summarized briefly in Section 5 along with some concluding observations.

2.0 Background Theory and Discussion

A general quadric in arbitrary position is represented algebraically as:

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz + 2Gx + 2Hy + 2Jz + K = 0 \qquad (1)$$

This equation can be written in matrix form:

$$pQp^T = 0$$

where

$$Q = \begin{pmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{pmatrix}$$

$$p = (x, y, z, 1)$$

The 4x4 symmetric matrix Q completely characterizes a given quadric; therefore we will refer to particular quadrics in terms of their corresponding matrices. For simplicity of expression, however, we will say "given a quadric Q ...." rather than the precise but more awkward "given a quadric whose 4x4 symmetric matrix is Q....". Note that the equation of a quadric surface, and therefore the matrix Q, is unique only up to constant multiples.

The natural quadrics are the sphere, the right circular cylinder, and the right circular cone. The general quadric surface is either a cylinder or cone lying over a conic section, or it is an ellipsoid, paraboloid, or hyperboloid. However a quadric may be degenerate and actually consist of one or two lower degree or lower dimensional shapes. The possibilities are:

- *a single plane:* if G, H, J, and K are the only non-zero terms in (1)
- *a pair of identical, parallel, or intersecting planes:* if (1) can be factored into two terms, each

of which is linear in x, y, and z
- *a single line:* if, for example, A and B are the only non-zero elements of Q
- *a single point:* if, for example, A, B, and C are the only non-zero elements of Q

In this paper, when we speak of "planar surfaces" we mean any of these possibilities *except* the single point. We shall see why the single point is excluded in the proof of Theorem 1 later in this section.

Given two quadrics Q_1 and Q_2, we can describe parametrically a family of quadrics (called the pencil of Q_1 and Q_2) as:

$$Q(\lambda_1, \lambda_2) = \lambda_1 Q_1 + \lambda_2 Q_2$$

where λ_1 and λ_2 are arbitrary real numbers, at least one of which is non-zero. The pencil is fundamental in the study of quadric intersections because any pair of quadrics in the same pencil have the same intersection curve. If we discover that the pencil of two quadrics contains a planar surface, for example, then the intersection of the given quadrics must be planar. This notion is formalized below and forms the basis of the algebraic analysis strategy used in this paper. Since we shall be studying intersections between non-degenerate natural quadrics, in this paper Q_1 and Q_2 will represent only spheres, right circular cylinders, and right circular cones. Furthermore, since our approach will be to seek planar surfaces in the pencil, we shall never be interested in $\lambda_1=0$ or $\lambda_2=0$. Therefore, recalling that constant multiples of quadrics represent the same surface, we can divide by λ_1, set $\lambda = -\lambda_2/\lambda_1$, and use a simpler single-parameter representation of the pencil:

$$Q(\lambda) = Q_1 - \lambda Q_2.$$

In order to establish the conditions under which a pair of quadric surfaces intersect degenerately, we need the following two theorems. Similar statements of and proofs for these theorems are given in [12]. We include them here for completeness.

Theorem 1: The intersection of two quadric surfaces is a planar curve (or a pair of planar curves) if and only if there is a planar surface in the pencil of the two quadric surfaces.

Proof: The pencil of two quadrics Q_1 and Q_2 is given parametrically by $Q(\lambda) = Q_1 - \lambda Q_2$. Clearly the intersection of Q_1 and Q_2 lies on every quadric surface in the pencil. Thus if there is a planar surface (or a pair of planar surfaces) in the pencil, the intersection curve must be a planar curve (or a pair of planar curves).

That there is always a planar surface in the pencil when the intersection is a planar curve (or a pair of planar curves) is not so clear. Let the intersection curve contain a conic C, and let P be the plane containing C. We first show that the pencil must contain a surface which contains P. Let $P_0=(x_0,y_0,z_0)$ be a point on P which is not on C and consider $\lambda_0 = Q_1(x_0,y_0,z_0)/Q_2(x_0,y_0,z_0)$. Then $Q(\lambda_0) = Q_1 - \lambda_0 Q_2$ contains both C and P_0 since $Q(\lambda_0)(x_0,y_0,z_0)=0$.[1] Thus the quadric surface $Q(\lambda_0)$ and the plane P intersect in a conic section plus one additional point. But this is impossible unless $Q(\lambda_0)$ actually contains P since a quadric surface and a plane always intersect in exactly one conic section with no additional points. Thus we have constructed a quadric surface in the pencil of Q_1 and Q_2 which contains a plane. But the only quadric surfaces which contain a plane are the plane

[1] If $Q_2(x_0,y_0,z_0)=0$, then just choose $Q(\lambda_0)=Q_2$.

itself or a pair of parallel or intersecting planes. *QED*. [2]

Theorem 2: A quadric surface represents a planar surface if and only if the rank of the 4x4 symmetric matrix which represents the surface is less than or equal to 2.

Proof: The proof follows directly from the general classification scheme for quadric surfaces presented in [6] (Table I, page 556). We observe there that all the singular planar cases (as summarized above) have rank 1 or 2.[3] *QED*.

The proof of Theorem 1 can easily be extended to the case where the conic section is just two points and the planar surface is a straight line. The proof fails, however, when the quadrics intersect in a single point, because it is not possible to conclude that there is a single point surface in the pencil. In fact, this is not always the case. For example, a unit sphere at the origin and the double plane $(z-1)^2=0$ are tangent at the point (0,0,1), but there is no single point surface in their pencil. It is therefore unclear whether the methods described in this paper can be used to detect all cases of single point tangencies between quadric surfaces. Two point tangencies are detected, however. We will see this clearly later.

Whether it is of value to detect all the situations under which two quadrics intersect in one or two isolated tangent points depends on the application. In solid modeling, such situations may indicate the presence of nonmanifold topologies on the boundary of the solid, and may therefore be of some interest. Whether the frequency with which this type of situation arises justifies the work necessary to detect all such cases is unclear. Since, however, detecting two-point tangencies will come for free with the analysis to be described here, we include these situations in our analysis. We choose not to attempt a characterization of all one-point tangencies, although establishing at least a partial such characterization seems plausible by systematically establishing the conditions on pairs of quadric surfaces which would force the pencil to contain a single-point surface. We note also that the natural quadric intersection algorithms described in [7] detect two-point tangencies without explicitly checking the expressions derived in sections 4.3, 4.5, and 4.6. They also detect all single point tangencies known to us, but there is, of course, no guarantee that they detect all such possible situations. (We shall amplify on some of these remarks in [8].)

2.1 Previous Work

The first reported technique for detecting conic sections in a computer-based implementation of quadric surface intersections was described by Levin [6]. This method was later extended by Sarraga [12]. As we do in the analysis described here, Sarraga searches the pencil looking for a planar surface. There are, however, fundamental differences in the two approaches. Given a pair of quadrics Q_1 and Q_2, Sarraga forms a numerical representation of their pencil $Q(\lambda)$ and computes the numerical value of λ which minimizes the rank of $Q(\lambda)$. If the resulting rank is less than or equal to two, then he manipulates $Q(\lambda)$ using another numerical method described by Dresden [1] to compute the equations of the planes containing the conic sections. Once he has these planes, he intersects them with one of the two original quadrics to find the resulting conic intersection curves. The tests and other computations required to carry out this analysis are well-defined mathematically, but require numerically sensitive calculations. In summary, Sarraga applies purely computer-based numerical methods to each pair of quadrics in the data base using sensitive numerical tests to detect planar intersections. Our method is based on a symbolic analysis of the six possible combinations of natural quadric surfaces, done once and for all and by hand. The results of this symbolic analysis are a simple

[2] The approach used in this proof is a standard technique in algebraic geometry (in fact, essentially the same argument is used in [9]), but we do not know of a good original reference for it.

[3] We note also that the single point case (imaginary cone) has rank 3. This is why the single point case is not included in this theorem.

set of robust geometric tests which are implemented on a computer to detect planar intersections between a given pair of natural quadrics [8].

Piegl [11] presents an approach based on a geometric construction for computing quadric surface intersection curves. He is given a pair of natural quadrics, each represented as a trimmed tensor product rational B-spline surface. He first numerically extracts from the B-spline description the geometric data describing the quadric (e.g., the center point and radius if it is a sphere), and he then analyzes this data geometrically to determine the type of intersection. To detect conics, he relies on a theorem of projective geometry which states that the intersection of two quadrics is planar if there exist four non-coplanar points common to the two quadrics such that the surfaces are tangent at two of them. In his paper he describes how to use this theorem to detect planar cylinder-cone intersections. The conditions he derives are considerably more complex than those derived here. Moreover, he gives no completeness argument. In fact, the theorem on which he bases his analysis is sufficient, but not necessary. It is unclear whether his algorithms are guaranteed to detect all planar intersections involving natural quadrics. It is also unclear whether two tangent point intersections are detected as they are in the method described here.

Farouki, Neff, and O'Connor [2] address the problem of automatically determining all degenerate intersections involving quadric surfaces of all types. The presence of degenerate intersection branches (whether they be lines, conics, cubics, or nodal or cuspidal quartics) is signalled by the vanishing of various polynomial expressions involving the quadric coefficients. When such a situation is detected, a multivariate polynomial factorization algorithm is invoked to isolate the various reducible components of the intersection. Again, the primary advantage of our method lies in its simplicity and in its ability to employ purely geometric reasoning without the need for multivariate factorization. The disadvantage is a lack of generality: the method of Farouki, Neff, and O'Connor works for arbitrary quadrics while ours must be tailored to individual pairs of quadric surface types.

Ocken, et. al. [9] use algebraic representations to parameterize intersections between general pairs of quadric surfaces. Their analysis is considerably more complex than ours, but this is, at least in part, because they are treating the general case rather than just the natural quadrics. More significantly, their analysis is numerical and must be performed at execution time for each pair of surfaces in the model. Furthermore, while some mention is made of configurations in which the intersection is planar, there is no comprehensive treatment of these situations. Thus there are configurations in which the intersection would degenerate into, say, a pair of ellipses, but for which their algorithm would parameterize the result as if it were a general degree four space curve. In contrast, our analysis is symbolic and performed only once, by hand rather than by computer, in order to characterize in a general fashion all configurations in which the intersection is planar. Our implementation [8] is purely geometric and is driven by the characterizations derived in this paper. There is some similarity, however, between their approach and ours. Both analyses begin by considering quadrics defined in canonical position, but generate results which are independent of position and orientation. The generality of Ocken, et. al. is achieved by studying invariants such as the eigenvalues of surfaces; our generality derives from the invariance of vector expressions under affine transformations.

O'Connor [10] determines parameterizations for the curve of intersection between an arbitrary pair of natural quadric surfaces using various geometric constructions and projections. As with Ocken, et. al. [9], there is no systematic attempt to detect degeneracies in order to represent the result explicitly as one or two (possibly degenerate) conic sections. An intersection which splits into a pair of ellipses would be parameterized as a general quartic space curve using O'Connors approach. However, O'Connor is able to use geometric constructions and projections to simplify a critical aspect of the intersection operation. The majority of the earlier approaches to the quadric surface intersection problem require solving a fourth degree equation in order to partition the parameter space of one of the quadric surfaces. O'Connor's approach, like that described in [7], requires solving only a quadratic equation. In fact, we rely on this aspect of the methods described in [7] for our treatment of isolated tangent points in [8].

Finally, Shene and Johnstone [13] describe a geometric approach for detecting planar intersections between pairs of natural quadric surfaces whose theoretical justification is established with purely geometric reasoning. They begin by demonstrating that two axial natural quadrics can have a conic intersection component only if their axes are coplanar. Given two such axial natural quadrics, they form two pairs of lines by cutting each quadric with the plane containing the axes. There are then a maximum of four points at which these two line pairs may intersect. Diagonals are the lines not contained on either quadric which connect pairs of these points. Portions of these diagonals which are in the interior of both quadrics are termed "potential segments". Shene and Johnstone define the "height" of a natural quadric above an arbitrary point in the interior of a potential segment as the distance from the point to the quadric along a line perpendicular to the axial plane. They then demonstrate that the intersection of the two quadrics contains a conic if and only if the heights to both surfaces from such a point are the same. Their algorithm mirrors this logic. If the axes of two quadrics are found to lie in the same plane, then the diagonals are computed, and potential segments are identified. For each potential segment, a point in the interior is generated and used in the height test. If the test is satisfied, the one or two points bounding the potential segment and an additional point are used to compute the parameters of the conic.

This method is markedly different from ours. Theirs is a procedural approach for the detection of conic components while ours is driven from an exhaustive (but surprisingly short) list of specific relative geometric configurations. The primary advantage of their method is that their procedural detection leads to a common geometric algorithm for axial quadrics. On the other hand, testing our conditions is more straightforward and computationally efficient than computing diagonals, identifying potential segments, and performing height tests. Once a conic component has been detected, they use the diagonal intersection points (which are the vertices of the conic) and an additional point on the conic to compute the parameters, presumably by forcing them to satisfy equations describing distance relationships. Our implementation is described in [8] where we either derive simple explicit formulas for the conic parameters or show how to construct geometric descriptions for the planes containing the conic sections.

3.0 Overview of the Method

We treat only the natural quadrics (i.e., sphere, right circular cylinder, and right circular cone) in this paper. For simplicity of expression, we shall use the term "cylinder" for "right circular cylinder" and "cone" for "right circular cone". While the analysis is certainly applicable to the other quadrics as well, the details get increasingly complex. Fully half of this paper is devoted to analyzing the intersection between two cones.

For each of the six possible combinations of pairs of natural quadric surfaces (sphere-sphere, sphere-cylinder, sphere-cone, cylinder-cylinder, cylinder-cone, and cone-cone), we seek to establish all possible sets of necessary and sufficient conditions so that their pencil contains a planar surface. The approach to this analysis is derived from Theorem 2. Then, because of Theorem 1, we will know that conic sections arise in these and only these cases. In theory, we could proceed as follows.

For each of the six combinations of pairs of natural quadric surface types:

(a) Derive the equation of a quadric of each type in general position.
(b) Write the resulting general equation of a quadric in their pencil.
(c) Determine conditions on the original two surfaces so that there is a planar surface in their pencil. These conditions are precisely those that force all 3x3 subdeterminants of the pencil matrix to be zero, thereby forcing this matrix to have rank less than or equal to 2.

For each pair of surface types, step (c) will result in sets of algebraic constraints which will give rise to planar intersections. These algebraic conditions can then be reinterpreted in terms of geometric invariants involving the two original quadrics. The value of this approach is that the geometric constraints have obvious physical interpretations, and they can therefore be detected with greater numerical reliability than can pure formal algebraic conditions.

Unfortunately the equations of a cylinder and a cone in general position are far too complex for this analysis. It was for this reason that the notions of *canonical* and *relative canonical* position were introduced in [3]. Briefly the idea is as follows. Without loss of generality, we can write the surface equations with respect to a coordinate system in which the surfaces have positions and orientations which are as simple as possible. This can be imagined as a two-step process. First we rigidly transform the two surfaces into a coordinate system in which one is described with the simplest equation possible. The other winds up in some general position and orientation with respect to this coordinate system. Next we rotate about an axis of this system which will not further alter the equation of the first quadric, but which will place the second in the simplest possible orientation with respect to the coordinate system. This has the effect of introducing as many zeros as possible into the surface matrices, thereby simplifying the algebra.

Table I

Canonical and Relative Canonical Position for the Natural Quadric Surfaces

SURFACE	CANONICAL POSITION	RELATIVE CANONICAL POSITION
sphere	Center at origin $x^2 + y^2 + z^2 - r^2 = 0$	(Arbitrary Position[4]; Center=(μ,υ,ω)) $(x-\mu)^2 + (y-\upsilon)^2 + (z-\omega)^2 - r^2 = 0$
cylinder	Axis is z-axis $x^2 + y^2 - r^2 = 0$	Axis=$(0,s,c)$; Point on Axis=(μ,υ,ω) $x^2 + c^2y^2 + s^2z^2 - 2scyz - 2\mu x - 2(\upsilon c^2 - sc\omega)y - 2(\omega s^2 - sc\upsilon)z + \mu^2 + c^2\upsilon^2 + s^2\omega^2 - 2sc\upsilon\omega - r^2 = 0$
cone	Vertex at origin; Axis is z-axis $x^2 + y^2 - E^2z^2 = 0$ NOTE: α is the cone half angle; E=tanα; and F=secα	Axis=$(0,s,c)$; Vertex=(μ,υ,ω) $x^2 + (1-s^2F^2)y^2 + (1-c^2F^2)z^2 - 2scF^2yz - 2\mu x - 2(\upsilon(1-s^2F^2) - sc\omega F^2)y - 2(\omega(1-c^2F^2) - sc\upsilon F^2)z + \mu^2 + \upsilon^2 + \omega^2 - F^2(s\upsilon + c\omega)^2 = 0$

Table I describes the canonical and relative canonical position for each of the three natural quadrics. (See [3] for derivations[5].) The axis vector used in the relative canonical position for cylinders and cones is assumed to be a unit vector; that is, $s^2+c^2=1$. For each of the six quadric surface type pairs, we put the most complex in canonical position, and the other in relative canonical position. (We order the quadrics from simple to complex as: sphere, cylinder, cone.) The actual strategy followed in this paper to establish the conditions for degenerate quadric surface intersections is then:

For each of the six combinations of natural quadric surface types:

(a) Derive the matrices of the two quadrics, the more complex in canonical position, and the

[4]The sphere is sufficiently simple that we will always consider it in arbitrary position.

[5]The equations shown here are actually slightly more complete than those developed in [3] as only the upper left 3x3 portion of the matrix Q was actually derived and used there.

other in relative canonical position.
(b) Write the resulting general matrix for a quadric in their pencil.
(c) Determine conditions on the original two surfaces so that there is a planar surface in their pencil. These conditions are precisely those that force all 3x3 subdeterminants of the pencil matrix to be zero, thereby forcing this matrix to have rank less than or equal to 2.

One more device to simplify the algebra will occasionally be available to us when dealing with a cylinder in relative canonical position. If we know that s (the y-component of the axis vector) is non-zero, then we know that the axis line will intersect the y=0 plane. Since the base point of the cylinder (μ,υ,ω) is arbitrary, we can, without loss of generality, choose μ and ω so that $\upsilon=0$. We will exploit this additional simplification whenever possible.

The analysis begins by forcing the upper left 3x3 subdeterminant to be zero. This leads to an equation in λ and therefore to a separate subcase for each distinct solution λ. We shall see that $\lambda=1$ is always a root. That this is true in general is proved in [3].

To denote the various 3x3 subdeterminants of $Q(\lambda)$, we use cofactor notation. For example, the upper left 3x3 subdeterminant is denoted as Cofactor Q_{44}. The dependence on λ is not shown in order to simplify the notation.

Finally we emphasize that we use canonical and relative canonical position only to perform the formal algebraic analysis. On the basis of this analysis, we shall deduce geometric constraints independent of any coordinate system which must be satisfied by a pair of natural quadrics in order for them to intersect degenerately. It is these geometric invariants (distances between points, angles between axes, etc.) which are then tested in the computer-based implementation. No transformations of any sort are required in the actual implementation. While the algebraic analysis is at times difficult, the final results (i.e., the geometric invariants to be tested in the computer implementation) are surprisingly simple.

4.0 Pairwise Analysis of Surface Intersections

In this section we derive necessary and sufficient conditions for degenerate intersections to arise between the six possible combinations of pairs of natural quadric surfaces. We proceed in order from the simplest to the most complicated cases. The flavor of our analysis can be obtained from reading the simpler cases. The most complicated case is cone-cone which takes up about half of the paper. A summary of all our results follows in Section 5.

4.1 Sphere-Sphere Intersections

That sphere-sphere intersections always result in planar curves can be established without resort to canonical and relative canonical position. Consider two spheres in arbitrary position:

$$Q_1 : x^2 + y^2 + z^2 - 2\mu_1 x - 2\upsilon_1 y - 2\omega_1 z + \mu_1^2 + \upsilon_1^2 + \omega_1^2 - r_1^2 = 0$$
$$Q_2 : x^2 + y^2 + z^2 - 2\mu_2 x - 2\upsilon_2 y - 2\omega_2 z + \mu_2^2 + \upsilon_2^2 + \omega_2^2 - r_2^2 = 0$$

$$Q_1 - Q_2 : 2(\mu_2 - \mu_1)x + 2(\upsilon_2 - \upsilon_1)y + 2(\omega_2 - \omega_1)z + \mu_1^2 + \upsilon_1^2 + \omega_1^2 - \mu_2^2 - \upsilon_2^2 - \omega_2^2 + r_2^2 - r_1^2 = 0$$

Since $Q_1 - Q_2$ is linear in x, y, and z, it is a plane. Therefore Theorem 1 tells us the intersection of two spheres is always planar. The intersection is a single tangent point if the distance between the centers is $|r_1-r_2|$ or (r_1+r_2), a circle if the distance between the centers is between these limits, and empty otherwise.

4.2 Sphere-Cylinder Intersections

Consider the cylinder in canonical position and the sphere in arbitrary position:

$$Q_{cyl} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_c^2 \end{pmatrix}$$

$$Q_{sph} = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ 0 & 1 & 0 & -\upsilon \\ 0 & 0 & 1 & -\omega \\ -\mu & -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r_s^2 \end{pmatrix}$$

The pencil of these surfaces is then:

$$Q(\lambda) = Q_{sph} - \lambda Q_{cyl} = \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & 1-\lambda & 0 & -\upsilon \\ 0 & 0 & 1 & -\omega \\ -\mu & -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r_s^2+\lambda r_c^2 \end{pmatrix}$$

We begin by forcing Cofactor Q_{44} to be zero:

$$Cofactor Q_{44} = \det \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1-\lambda)^2 = 0.$$

Thus we conclude that $\lambda=1$ is the only possibility, and we need to consider:

$$Q(1) = \begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & -\upsilon \\ 0 & 0 & 1 & -\omega \\ -\mu & -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r_s^2+r_c^2 \end{pmatrix}$$

Forcing Cofactor Q_{11} and Cofactor Q_{22} to be zero gives us:

$$Cofactor Q_{11} = \det \begin{vmatrix} 0 & 0 & -\upsilon \\ 0 & 1 & -\omega \\ -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r_s^2+r_c^2 \end{vmatrix} = -\upsilon^2 = 0$$

and

$$Cofactor Q_{22} = \det\begin{vmatrix} 0 & 0 & -\mu \\ 0 & 1 & -\omega \\ -\mu & -\omega & \mu^2+\upsilon^2+\omega^2-r_s^2+r_c^2 \end{vmatrix} = -\mu^2 = 0.$$

Collecting these conditions, we conclude that μ and υ must both be zero in order for the intersection to be planar. Geometrically this means that the center of the sphere must lie on the *z*-axis, that is, on the axis of the cylinder. Since Rank($Q(1)$)=2 when μ=υ=0, we conclude from Theorems 1 and 2 that the intersection is planar if and only if the center of the sphere lies on the axis of the cylinder. In this case there are three possibilities: two circles in parallel planes (if $r_s>r_c$), a single tangent circle of intersection (if $r_s=r_c$), or no real intersection at all (if $r_s<r_c$). Note that there is also no real intersection if the distance between the sphere center and the cylinder axis is greater than (r_c+r_s) or less than $|r_c-r_s|$.

4.3 Sphere-Cone Intersections

Consider the cone in canonical position and the sphere in arbitrary position:

$$Q_{con} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -E^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_{sph} = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ 0 & 1 & 0 & -\upsilon \\ 0 & 0 & 1 & -\omega \\ -\mu & -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r^2 \end{pmatrix}$$

The pencil of these surfaces is then:

$$Q(\lambda) = Q_{sph} - \lambda Q_{con} = \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & 1-\lambda & 0 & -\upsilon \\ 0 & 0 & 1+\lambda E^2 & -\omega \\ -\mu & -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r^2 \end{pmatrix}$$

We begin by forcing Cofactor Q_{44} to be zero:

$$Cofactor Q_{44} = \det\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1+\lambda E^2 \end{vmatrix} = (1-\lambda)^2(1+\lambda E^2) = 0.$$

Thus there are two possibilities: λ=1 and λ=-1/E^2.

Case 1: λ=1

Using λ=1 we have:

$$Q(1)=\begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & -\upsilon \\ 0 & 0 & 1+E^2 & -\omega \\ -\mu & -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r^2 \end{pmatrix}$$

The analysis is now similar to the sphere-cylinder case. Forcing Cofactor Q_{11} and Cofactor Q_{22} to be zero gives us:

$$Cofactor Q_{11}=\det\begin{vmatrix} 0 & 0 & -\upsilon \\ 0 & 1+E^2 & -\omega \\ -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r^2 \end{vmatrix}=-\upsilon^2\left(1+E^2\right)=0$$

and

$$Cofactor Q_{22}=\det\begin{vmatrix} 0 & 0 & -\mu \\ 0 & 1+E^2 & -\omega \\ -\mu & -\omega & \mu^2+\upsilon^2+\omega^2-r^2 \end{vmatrix}=-\mu^2\left(1+E^2\right)=0.$$

Collecting these conditions, we conclude that μ and υ must both be zero in order for the intersection to be planar. Geometrically this means that the center of the sphere must lie on the axis of the cone. The possibilities are a single tangent circle, two circles in parallel planes on the same half of the cone, a circle plus another single point of intersection at the vertex, two circles in parallel planes on opposite halves of the cone, or no intersection at all [8].

Case 2: λ=-1/E^2

Using $\lambda=-1/E^2$ we have:

$$Q\left(-1/E^2\right)=\begin{pmatrix} \left(E^2+1\right)/E^2 & 0 & 0 & -\mu \\ 0 & \left(E^2+1\right)/E^2 & 0 & -\upsilon \\ 0 & 0 & 0 & -\omega \\ -\mu & -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r^2 \end{pmatrix}$$

Forcing Cofactor Q_{11} to be zero gives us:

$$Cofactor Q_{11}=\det\begin{vmatrix} \left(E^2+1\right)/E^2 & 0 & -\upsilon \\ 0 & 0 & -\omega \\ -\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-r^2 \end{vmatrix}=\frac{-\left(E^2+1\right)\omega^2}{E^2}=0.$$

This implies that $\omega=0$, that is, that the sphere center must lie in the $z=0$ plane of the cone's canonical coordinate system. In coordinate system independent terms, this means that the sphere center must lie in the plane determined by the cone vertex and the cone axis vector. To complete the set of constraints corresponding to case 2, we force Cofactor Q_{33} to be zero (recall we now know $\omega=0$):

$$Cofactor Q_{33} = \det\begin{vmatrix} (E^2+1)/E^2 & 0 & -\mu \\ 0 & (E^2+1)/E^2 & -\upsilon \\ -\mu & -\upsilon & \mu^2+\upsilon^2-r^2 \end{vmatrix}$$

$$= \frac{(E^2+1)}{E^2}\left[\frac{(E^2+1)}{E^2}(\mu^2+\upsilon^2-r^2)-(\mu^2+\upsilon^2)\right]=0$$

$$\therefore (E^2+1)(\mu^2+\upsilon^2-r^2)=E^2(\mu^2+\upsilon^2)$$

$$\therefore (E^2+1)r^2=\mu^2+\upsilon^2$$

The quantity $\sqrt{(\mu^2+\upsilon^2)}$ is simply the distance, *d*, between the sphere center and the cone vertex (since $\omega=0$). Taking square roots and recalling that $E=\tan\alpha$, we find that this constraint can be rewritten in terms of geometric invariants as:

$$r = d\cos\alpha.$$

As illustrated in Figure 1, these two constraints force the sphere to have precisely two real points of tangency with the cone.

4.4 Cylinder-Cylinder Intersections

The first cylinder in canonical position is described by:

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_1^2 \end{pmatrix}$$

The second in relative canonical position is then:

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ 0 & c^2 & -sc & sc\omega-\upsilon c^2 \\ 0 & -sc & s^2 & sc\upsilon-\omega s^2 \\ -\mu & sc\omega-\upsilon c^2 & sc\upsilon-\omega s^2 & \mu^2+c^2\upsilon^2+s^2\omega^2-2sc\upsilon\omega-r_2^2 \end{pmatrix}$$

Thus the pencil is

$$Q(\lambda)=Q_2-\lambda Q_1=\begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & c^2-\lambda & -sc & sc\omega-\upsilon c^2 \\ 0 & -sc & s^2 & sc\upsilon-\omega s^2 \\ -\mu & sc\omega-\upsilon c^2 & sc\upsilon-\omega s^2 & \mu^2+c^2\upsilon^2+s^2\omega^2-2sc\upsilon\omega-r_2^2+\lambda r_1^2 \end{pmatrix}$$

As usual, we begin by forcing Cofactor Q_{44} to be zero:

$$CofactorQ_{44} = \det\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & c^2-\lambda & -sc \\ 0 & -sc & s^2 \end{vmatrix} = -(1-\lambda)\lambda s^2 = 0\,.$$

There are three possibilities: $\lambda=0$, $\lambda=1$, and $s=0$. We will consider $s=0$ before $\lambda=1$ so that we can assume $s\neq0$ when treating $\lambda=1$. This will simplify the analysis.

Case 1: λ=0

Since $Q(0)=Q_2$ and Q_2 is a cylinder, this possibility does not lead to a planar surface.

Case 2: s=0

When $s=0$, $c=1$ since the unit axis vector of Q_2 is $(0,s,c)$. Therefore this constraint tells us that the axes of the two cylinders must be parallel. When this is the case, the intersection is either one tangent line if the distance between the axes is (r_1+r_2) or $|r_1\text{-}r_2|$; two parallel lines if the distance between the axes is between these two limits; or empty otherwise.

Case 3: λ=1, s≠0

Recall that Q_2 is in relative canonical position, and therefore its axis is $(0,s,c)$. Since $s\neq0$, we can set $\upsilon=0$ as noted in Section 3.0. Assuming $\upsilon=0$ and using $\lambda=1$, we find:

$$Q(1) = \begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & -s^2 & -sc & sc\omega \\ 0 & -sc & s^2 & -\omega s^2 \\ -\mu & sc\omega & -\omega s^2 & \mu^2+s^2\omega^2-r_2^2+r_1^2 \end{pmatrix}$$

Forcing Cofactor Q_{14} to be zero gives us:

$$CofactorQ_{14} = \det\begin{vmatrix} 0 & -s^2 & -sc \\ 0 & -sc & s^2 \\ -\mu & sc\omega & -\omega s^2 \end{vmatrix} = \mu s^2\left(s^2+c^2\right) = \mu s^2 = 0\,.$$

This tells us that $\mu=0$ since $s\neq0$ by assumption. The base point of cylinder Q_2 is therefore $(0,0,\omega)$. Since this point is on the z-axis, this constraint tells us that the cylinder axes intersect. To complete this set of constraints, we next force Cofactor Q_{11} to be zero:

$$CofactorQ_{11} = \det\begin{vmatrix} -s^2 & -sc & sc\omega \\ -sc & s^2 & -\omega s^2 \\ sc\omega & -\omega s^2 & s^2\omega^2+r_1^2-r_2^2 \end{vmatrix} = 0\,.$$

If we multiply the second row by ω, add it to the third row, and then expand by cofactors of the bottom row, we obtain:

$$
CofactorQ_{11} = \det\begin{vmatrix} -s^2 & -sc & sc\omega \\ -sc & s^2 & -\omega s^2 \\ 0 & 0 & r_1^2 - r_2^2 \end{vmatrix} = -(r_1^2 - r_2^2)s^2(s^2 + c^2) = -(r_1^2 - r_2^2)s^2 = 0.
$$

Again since $s \neq 0$ by assumption, this tells us that the radii of the cylinders must be the same. In summary, then, the result of case 3 is the well-known fact that two cylinders will intersect in planar curves (namely, two ellipses) if they have the same radius and if their axes intersect.

4.5 Cylinder-Cone Intersections

Consider the cone in canonical position and the cylinder in relative canonical position:

$$
Q_{con} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -E^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
Q_{cyl} = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ 0 & c^2 & -sc & sc\omega - \upsilon c^2 \\ 0 & -sc & s^2 & sc\upsilon - \omega s^2 \\ -\mu & sc\omega - \upsilon c^2 & sc\upsilon - \omega s^2 & \mu^2 + c^2\upsilon^2 + s^2\omega^2 - 2sc\upsilon\omega - r^2 \end{pmatrix}
$$

The pencil of these surfaces is then:

$$
Q(\lambda) = Q_{cyl} - \lambda Q_{con} = \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & c^2 - \lambda & -sc & sc\omega - \upsilon c^2 \\ 0 & -sc & s^2 + \lambda E^2 & sc\upsilon - \omega s^2 \\ -\mu & sc\omega - \upsilon c^2 & sc\upsilon - \omega s^2 & \mu^2 + c^2\upsilon^2 + s^2\omega^2 - 2sc\upsilon\omega - r^2 \end{pmatrix}
$$

We begin by forcing Cofactor Q_{44} to be zero:

$$
CofactorQ_{44} = \det\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & c^2 - \lambda & -sc \\ 0 & -sc & s^2 + \lambda E^2 \end{vmatrix} = \lambda(1-\lambda)(-\lambda E^2 + c^2E^2 - s^2) = 0.
$$

Thus there are three possibilities: $\lambda=0$, $\lambda=1$, and $\lambda=(c^2E^2-s^2)/E^2$.

Case 1: $\lambda=0$

Since $Q(0)=Q_{cyl}$, this possibility does not lead to a planar surface.

Case 2: $\lambda=1$

Using $\lambda=1$ we write:

$$Q(1)=\begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & -s^2 & -sc & sc\omega-\upsilon c^2 \\ 0 & -sc & s^2+E^2 & sc\upsilon-\omega s^2 \\ -\mu & sc\omega-\upsilon c^2 & sc\upsilon-\omega s^2 & \mu^2+c^2\upsilon^2+s^2\omega^2-2sc\upsilon\omega-r^2 \end{pmatrix}$$

Forcing Cofactor Q_{14} to be zero:

$$CofactorQ_{14}=\det\begin{vmatrix} 0 & -s^2 & -sc \\ 0 & -sc & s^2+E^2 \\ -\mu & sc\omega-\upsilon c^2 & sc\upsilon-\omega s^2 \end{vmatrix}=-\mu s^2\left(E^2+s^2+c^2\right)=-\mu s^2\left(E^2+1\right)=0.$$

We therefore have two subcases to consider: s=0 and μ=0.

Case 2a: λ=1, s=0 (hence c=1)

Since s=0 and c=1, the cylinder axis is (0,0,1). Geometrically this means that the cylinder and cone axes must be parallel. Moreover, $Q(1)$ simplifies to:

$$Q(1)=\begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & -\upsilon \\ 0 & 0 & E^2 & 0 \\ -\mu & -\upsilon & 0 & \mu^2+\upsilon^2-r^2 \end{pmatrix}$$

Forcing Cofactor Q_{11} and Q_{22} to be zero then completes the analysis of this subcase:

$$CofactorQ_{11}=\det\begin{vmatrix} 0 & 0 & -\upsilon \\ 0 & E^2 & 0 \\ -\upsilon & 0 & \mu^2+\upsilon^2-r^2 \end{vmatrix}=-\upsilon^2E^2=0$$

$$CofactorQ_{22}=\det\begin{vmatrix} 0 & 0 & -\mu \\ 0 & E^2 & 0 \\ -\mu & 0 & \mu^2+\upsilon^2-r^2 \end{vmatrix}=-\mu^2E^2=0.$$

We conclude that μ=υ=0. The base point on the cylinder is therefore (0,0,ω), i.e., a point on the z-axis. Thus Case 2a tells us that the intersection of a cylinder and a cone is planar when their axes coincide. The resulting curve is always two circles in parallel planes, one on each half of the cone.

Case 2b: λ=1, μ=0, s≠0

Since s≠0, we can again set υ=0 without loss of generality. The cylinder base point is then (0,0,ω), i.e., a point on the z-axis. This means that the two axes intersect in a point at distance ω from the cone vertex. These observations allow us to simplify $Q(1)$ as follows:

$$Q(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -s^2 & -sc & sc\omega \\ 0 & -sc & s^2+E^2 & -\omega s^2 \\ 0 & sc\omega & -\omega s^2 & s^2\omega^2 - r^2 \end{pmatrix}$$

Now forcing Cofactor Q_{11} to be zero gives us:

$$Cofactor Q_{11} = \det \begin{vmatrix} -s^2 & -sc & sc\omega \\ -sc & s^2+E^2 & -\omega s^2 \\ sc\omega & -\omega s^2 & s^2\omega^2 - r^2 \end{vmatrix} = 0.$$

We multiply the second row by ω, add it to the third row, and then expand by cofactors of the third row to get:

$$Cofactor Q_{11} = -\omega^2 E^2 s^2 (s^2 + c^2) + r^2 s^2 (s^2 + E^2 + c^2) = 0.$$

Since $E=\tan\alpha$, $s\neq 0$, and $(s^2+c^2)=1$, we conclude:

$$r = \omega \sin\alpha .$$

From this analysis of case 2b, we deduce that the intersection will consist of planar curves if (i) the axes intersect, and (ii) $r=\omega\sin\alpha$, where ω is the distance from the cone vertex to the point at which the axes intersect. Note that these constraints are independent of the angle between the axes. The three possible cases are illustrated in Figure 2: (a) two intersecting ellipses, (b) an ellipse and a ruling shared tangentially (when the angle between the axes is the same as the cone half-angle), and (c) two ellipses on opposite halves of the cone. Notice that $\omega\sin\alpha$ is the distance between the point where the axes intersect and the surface of the cone. Of course, r is also the distance between the point where the axes intersect and the surface of the cylinder. Our constraint requires these two distances to be equal. We shall expand on this observation in Section 5.

Case 3: $\lambda=(c^2E^2-s^2)/E^2$

We can assume that $s\neq 0$ since $s=0$ implies that $c=1$ which in turn implies that $\lambda=1$, and we have already analyzed this possibility in case 2a. Since $s\neq 0$, we can set $\upsilon=0$, use the given value of λ, and the pencil matrix becomes:

$$Q\left(\lambda = \frac{c^2E^2 - s^2}{E^2}\right) = \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & s^2/E^2 & -sc & sc\omega \\ 0 & -sc & c^2E^2 & -\omega s^2 \\ -\mu & sc\omega & -\omega s^2 & \mu^2 + s^2\omega^2 - r^2 \end{pmatrix}$$

Forcing Cofactor Q_{34} to be zero gives us:

$$Cofactor Q_{34} = \det \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & s^2/E^2 & -sc \\ -\mu & sc\omega & -\omega s^2 \end{vmatrix} = (1-\lambda)\omega s^2 (c^2 - s^2/E^2) = (1-\lambda)\omega s^2 \lambda = 0.$$

Since we have already studied and/or ruled out λ=0, λ=1, and s=0, we need only pursue ω=0. This allows us to simplify the pencil matrix to:

$$Q\left(\lambda=\frac{c^2E^2-s^2}{E^2}\right)=\begin{pmatrix}1-\lambda & 0 & 0 & -\mu\\ 0 & s^2/E^2 & -sc & 0\\ 0 & -sc & c^2E^2 & 0\\ -\mu & 0 & 0 & \mu^2-r^2\end{pmatrix}$$

Forcing Cofactor Q_{33} to be zero gives us:

$$Cofactor Q_{33}=\det\begin{vmatrix}1-\lambda & 0 & -\mu\\ 0 & s^2/E^2 & 0\\ -\mu & 0 & \mu^2-r^2\end{vmatrix}=s^2/E^2\left[(1-\lambda)(\mu^2-r^2)-\mu^2\right]=0.$$

Since by assumption, s≠0, we know the quantity in brackets must vanish. Substituting the current value of λ, doing some algebraic manipulation, and invoking some trigonometric identities, we arrive at:

$$\frac{r^2}{\mu^2}=1-\frac{\sin^2\alpha}{\sin^2\theta}$$

where α is the cone half-angle, and θ is the angle between the axis vectors. In order to apply this test without the need for transformations, we need an invariant characterization of μ. But that is straightforward. Recall our cylinder has base point (μ,0,0) and axis vector (0,s,c), with s≠0. Thus it is clear that the cylinder and cone axes are skew with the cylinder base point as the point of closest approach. Hence μ is simply the distance between the two axes. This configuration results in the cylinder and the cone just touching in two distinct points of tangency as shown in Figure 3.

4.6 Cone-Cone Intersections

The first cone in canonical position is described by:

$$Q_1=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -E_1^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

The second in relative canonical position is then:

$$Q_2=\begin{pmatrix}1 & 0 & 0 & -\mu\\ 0 & 1-s^2F_2^2 & -scF_2^2 & sc\omega F_2^2-\upsilon(1-s^2F_2^2)\\ 0 & -scF_2^2 & 1-c^2F_2^2 & sc\upsilon F_2^2-\omega(1-c^2F_2^2)\\ -\mu & sc\omega F_2^2-\upsilon(1-s^2F_2^2) & sc\upsilon F_2^2-\omega(1-c^2F_2^2) & \mu^2+\upsilon^2+\omega^2-F_2^2(s\upsilon+c\omega)^2\end{pmatrix}$$

Thus the pencil is

$$Q(\lambda) = Q_2 - \lambda Q_1$$
$$= \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & 1-\lambda-s^2F_2^2 & -scF_2^2 & sc\omega F_2^2 - \upsilon(1-s^2F_2^2) \\ 0 & -scF_2^2 & 1+\lambda E_1^2 - c^2F_2^2 & sc\upsilon F_2^2 - \omega(1-c^2F_2^2) \\ -\mu & sc\omega F_2^2 - \upsilon(1-s^2F_2^2) & sc\upsilon F_2^2 - \omega(1-c^2F_2^2) & \mu^2+\upsilon^2+\omega^2 - F_2^2(s\upsilon+c\omega)^2 \end{pmatrix}.$$

As usual, we begin by forcing Cofactor Q_{44} to be zero:

$$CofactorQ_{44} = \det\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda-s^2F_2^2 & -scF_2^2 \\ 0 & -scF_2^2 & (1+\lambda E_1^2)-c^2F_2^2 \end{vmatrix} = 0$$
$$= (1-\lambda)\left\{E_1^2\lambda^2 - \left[(E_1^2+E_2^2) - s^2(E_1^2+E_2^2+E_1^2E_2^2+1)\right]\lambda + E_2^2\right\} = 0. \tag{2}$$

There are therefore two possibilities: λ=1 and λ is a root of the polynomial in the braces.

Case 1: λ=1

Using λ=1 we have:

$$Q(1) = \begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & -s^2F_2^2 & -scF_2^2 & sc\omega F_2^2 - \upsilon(1-s^2F_2^2) \\ 0 & -scF_2^2 & F_1^2 - c^2F_2^2 & sc\upsilon F_2^2 - \omega(1-c^2F_2^2) \\ -\mu & sc\omega F_2^2 - \upsilon(1-s^2F_2^2) & sc\upsilon F_2^2 - \omega(1-c^2F_2^2) & \mu^2+\upsilon^2+\omega^2 - F_2^2(s\upsilon+c\omega)^2 \end{pmatrix}$$

Forcing Cofactor Q_{14} to be zero gives us:

$$CofactorQ_{14} = \det\begin{vmatrix} 0 & -s^2F_2^2 & -scF_2^2 \\ 0 & -scF_2^2 & F_1^2 - c^2F_2^2 \\ -\mu & sc\omega F_2^2 - \upsilon(1-s^2F_2^2) & sc\upsilon F_2^2 - \omega(1-c^2F_2^2) \end{vmatrix} = \mu s^2F_1^2F_2^2 = 0.$$

We therefore have two subcases to consider: s=0 and μ=0.

Case 1a: λ=1, s=0 (hence c=1)

Since s=0 and c=1, the axis vector of the second cone is (0,0,1); i.e., it is parallel to the axis vector of the first cone. Moreover $Q(1)$ simplifies further to:

$$Q(1) = \begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & -\upsilon \\ 0 & 0 & F_1^2 - F_2^2 & -\omega(1-F_2^2) \\ -\mu & -\upsilon & -\omega(1-F_2^2) & \mu^2+\upsilon^2+\omega^2-F_2^2\omega^2 \end{pmatrix}$$

Forcing Cofactor Q_{11} and Cofactor Q_{22} to be zero then gives us:

$$Cofactor Q_{11} = \det\begin{vmatrix} 0 & 0 & -\upsilon \\ 0 & F_1^2 - F_2^2 & -\omega(1-F_2^2) \\ -\upsilon & -\omega(1-F_2^2) & \mu^2+\upsilon^2+\omega^2-F_2^2\omega^2 \end{vmatrix} = \upsilon^2(F_2^2 - F_1^2) = 0$$

$$Cofactor Q_{22} = \det\begin{vmatrix} 0 & 0 & -\mu \\ 0 & F_1^2 - F_2^2 & -\omega(1-F_2^2) \\ -\mu & -\omega(1-F_2^2) & \mu^2+\upsilon^2+\omega^2-F_2^2\omega^2 \end{vmatrix} = \mu^2(F_2^2 - F_1^2) = 0\,.$$

This leads to two subcases for case 1a: $F_1=F_2$ or $\mu=\upsilon=0$. Both of these conditions may, of course, hold simultaneously, in which case we get a single circle of intersection as noted in the discussion of Case 1a(i). Additionally we may have $\omega=0$, in which case the two cones are identical.

Case 1a(i): λ=1, s=0 (hence c=1), $F_1=F_2$

Since $F_1=F_2$, the cone half-angles are the same; since $s=0$ the cones also have parallel axes. These constraints tell us that the two cones are translates of each other. Notice that when $F_1=F_2$, all the entries of $Q(1)$ are zero except those in the last row and last column. Therefore $Q(1)$ is a single plane. These cones will intersect in a single real hyperbola, a single real ellipse (or circle), or a single real ruling shared tangentially, depending on the relative position of the vertices [8]; see Figure 4.

Case 1a(ii): λ=1, s=0 (hence c=1), $\mu=\upsilon=0$, $F_1 \neq F_2$

Since $\mu=\upsilon=0$ and the cones have parallel axes, we conclude that the cones have the same axis. Notice that when $\mu=\upsilon=0$, all the entries of $Q(1)$ are zero except those in the lower right 2x2 submatrix. This clearly makes the rank of $Q(1) \leq 2$ and therefore $Q(1)$ is a planar surface. (It is actually a pair of parallel planes of constant z.) If $\omega \neq 0$, the cones intersect in a pair of circles; otherwise they intersect in a single real point, namely their common vertex.

Case 1b: λ=1, μ=0, s≠0

Since $\mu=0$, the axis of the second cone lies entirely in the x=0 plane. (Its vertex is $(0,\upsilon,\omega)$, and its unit axis vector is $(0,s,c)$.) Since we also have $s \neq 0$, we know the axes of the two cones intersect. Because $\mu=0$, all elements in the first row and first column of $Q(1)$ are zero. This means that the only 3x3 subdeterminant we need to examine is Cofactor Q_{11}.

$$Cofactor Q_{11} = \det\begin{vmatrix} -s^2F_2^2 & -scF_2^2 & sc\omega F_2^2 - \upsilon(1-s^2F_2^2) \\ -scF_2^2 & F_1^2 - c^2F_2^2 & sc\upsilon F_2^2 - \omega(1-c^2F_2^2) \\ sc\omega F_2^2 - \upsilon(1-s^2F_2^2) & sc\upsilon F_2^2 - \omega(1-c^2F_2^2) & \upsilon^2+\omega^2-F_2^2(s\upsilon+c\omega)^2 \end{vmatrix} = 0$$

We multiply the first row by υ, the second row by ω, and add the results to the third row. Then we multiply the first column by υ, the second column by ω, and add the results to the third column. Finally we expand by cofactors of the last row to obtain:

$$CofactorQ_{11} = -\upsilon\left[sc\omega\left(1-F_1^2\right)F_2^2 + \upsilon F_1^2 - \upsilon c^2 F_2^2\right] + \omega\left(1-F_1^2\right)\left[s^2F_2^2\omega\left(1-F_1^2\right) - \upsilon scF_2^2\right] + s^2F_1^2F_2^2\left[\upsilon^2 + \omega^2\left(1-F_1^2\right)\right] = 0.$$

Gathering the coefficients of υ^2, $\upsilon\omega$, and ω^2, replacing c^2 by $(1-s^2)$, and using $F^2=E^2+1$, we obtain:

$$\left[s^2E_1^2\left(E_2^2+1\right) + \left(E_2^2 - E_1^2\right)\right]\upsilon^2 + 2scE_1^2\left(E_2^2+1\right)\upsilon\omega - s^2E_1^2\left(E_2^2+1\right)\omega^2 = 0. \quad (3)$$

This is a rather complex relationship to understand geometrically, yet certain conclusions can be drawn immediately. For example, if $\upsilon=0$, then we know the equation can only be satisfied if $\omega=0$ since $s\neq0$. This says that if the axes intersect at a vertex, the result is planar if and only if the vertices coincide.

Fortunately we need not try to understand this relationship completely as it stands since it can be simplified dramatically. Recall that in this case, we know the cone axes intersect in a distinct point. We shall denote this point as I. We begin our simplification of (3) by considering it to be a homogeneous quadratic equation in ω:

$$C\upsilon^2 + B\upsilon\omega + A\omega^2 = 0.$$

Our analysis then proceeds in four steps:
(1) Compute the discriminant of this equation.
(2) Apply the quadratic formula to solve for ω in terms of υ.
(3) Derive equivalent expressions for υ and ω in terms of d_1 and d_2, the distances between I and the respective cone vertices using geometry and trigonometry.
(4) Substitute the expressions for υ and ω derived in step (3) into the formula relating υ and ω derived in step (2).

To begin, we compute:

$$B^2 = 4s^2c^2E_1^4\left(E_2^2+1\right)^2 = 4s^2\left(1-s^2\right)E_1^4F_2^4 = 4s^2E_1^4F_2^4 - 4s^4E_1^4F_2^4$$

$$-4AC = 4s^4E_1^4\left(E_2^2+1\right)^2 + 4s^2E_1^2\left(E_2^2+1\right)\left(E_2^2-E_1^2\right) = 4s^4E_1^4F_2^4 + 4s^2E_1^2F_2^2\left(E_2^2-E_1^2\right)$$

$$B^2-4AC = 4s^2E_1^2F_2^2\left[E_1^2F_2^2 + \left(E_2^2-E_1^2\right)\right] = 4s^2E_1^2F_2^2\left[E_1^2\left(F_2^2-1\right)+E_2^2\right] = 4s^2E_1^2F_2^2\left[E_1^2E_2^2+E_2^2\right] = 4s^2E_1^2E_2^2F_1^2F_2^2.$$

Remarkably the quantity $(B^2\text{-}4AC)$ is a perfect square! This allows us to proceed in a very straightforward way to step (2). Using the quadratic formula, we express ω in terms of υ:

$$\omega = \left(\frac{-2scE_1^2(E_2^2+1) \pm 2sE_1E_2F_1F_2}{-2s^2E_1^2(E_2^2+1)} \right) \upsilon$$
$$= \left(\frac{c}{s} \pm \frac{E_2F_1}{sE_1F_2} \right) \upsilon.$$

Since

$$\frac{E}{F} = \frac{\tan\alpha}{\sec\alpha} = \sin\alpha$$

we get

$$\omega = \left(\frac{\cos\theta \pm \sin\alpha_2 / \sin\alpha_1}{\sin\theta} \right) \upsilon .$$

Before proceeding on to step (3), observe that we can always choose the coordinate axes so that the vertex $V_2=(0,\upsilon,\omega)$ lies in the first quadrant of the *yz*-plane; that is, we can assume that υ and ω are non-negative. Now we have two cases to consider with respect to the location of the axis intersection point *I*: it can lie either above or below the *y*-axis. (See Figure 5.) From Figure 5a, we see that if *I* lies on the positive *z*-axis, we can derive the following relationships for υ and ω:

$$\upsilon = d_2 \sin\theta$$
$$\omega = d_1 + d_2 \cos\theta .$$

From Figure 5b, we find that the following relationships hold when *I* lies on the negative z-axis:

$$\upsilon = d_2 \sin\theta$$
$$\omega = -d_1 + d_2 \cos\theta .$$

We can now substitute these expressions into the equation obtained in step (2) which expresses ω in terms of υ:

$$\pm d_1 + d_2 \cos\theta = \left(\frac{\cos\theta \pm \sin\alpha_2 / \sin\alpha_1}{\sin\theta} \right) d_2 \sin\theta$$
$$\therefore \pm d_1 = \left(\pm \frac{\sin\alpha_2}{\sin\alpha_1} \right) d_2 .$$

Since d_1 and d_2 are distances, they must both be positive. Furthermore, since the cone half-angles α_1 and α_2 are acute, their sines are positive. Therefore we need only consider the positive portion of the $\pm$ signs, and we arrive finally at the simple equation:

$$d_1 \sin\alpha_1 = d_2 \sin\alpha_2 .$$

This constraint has obvious geometric significance and can be tested easily and reliably in a computer-based implementation. Note the resemblance between this constraint and that developed for the cylinder-cone case (Section 4.5, Case 2b). The expression $d\sin\alpha$ represents the distance to the surface of the cone from a point on the axis a distance d from the vertex. Thus we see that if the axes of the cones intersect at a point equidistant from both surfaces, then the intersection is planar. Various pairs of possibly degenerate conics may arise. Some examples are illustrated in Figure 6(a-c).

Case 2: λ=<root of polynomial in braces in Equation (2)>≠1

We are not so lucky with the polynomial in braces in (2) as we were with (3). That is, the discriminant of (2) is apparently not a perfect square. Rather than solve directly for λ and carry complicated square roots around in subsequent equations, we finesse the problem by deriving constraints based on other considerations. To wit:

- Since $Q_{11}=(1-\lambda)\neq 0$ and $Q_{21}=Q_{31}=0$, we seek conditions which force the second and third rows of $Q(\lambda)$ to be dependent because the rank of $Q(\lambda)$ must be no greater than 2. This will generate two constraints, namely the vanishing of a pair of 2x2 determinants.
- Next we force all remaining 3x3 determinants to vanish. This will yield four additional constraints: Cofactor Q_{22}=Cofactor Q_{23}=Cofactor Q_{32}=Cofactor Q_{33}=0.

We begin by deriving constraints which force the second and third rows to be dependent. That is, we require:

$$rank\begin{pmatrix} 1-\lambda-s^2F_2^2 & -scF_2^2 & sc\omega F_2^2-\upsilon\left(1-s^2F_2^2\right) \\ -scF_2^2 & 1+\lambda E_1^2-c^2F_2^2 & sc\upsilon F_2^2-\omega\left(1-c^2F_2^2\right) \end{pmatrix}\leq 1.$$

We simplify this matrix by multiplying the first column by υ, the second column by ω, and adding the result to the third column:

$$rank\begin{pmatrix} 1-\lambda-s^2F_2^2 & -scF_2^2 & -\lambda\upsilon \\ -scF_2^2 & 1+\lambda E_1^2-c^2F_2^2 & \lambda E_1^2\omega \end{pmatrix}\leq 1.$$

Now any pair of 2x2 determinants must vanish. We choose to consider columns 1 and 3 and columns 2 and 3. First columns 1 and 3:

$$\det\begin{vmatrix} 1-\lambda-s^2F_2^2 & -\lambda\upsilon \\ -scF_2^2 & \lambda E_1^2\omega \end{vmatrix}=\lambda\left[\left(1-\lambda-s^2F_2^2\right)E_1^2\omega-scF_2^2\upsilon\right]=0.$$

As we have noted several times before, we can ignore $\lambda=0$ since that does not lead to a planar surface. This equation therefore leads us to consider the following cases:

- $\omega=0$ with one of the following:
  - $s=0$
  - $c=0$
  - $\upsilon=0$
- $$\lambda=\frac{\left(1-s^2F_2^2\right)E_1^2\omega-scF_2^2\upsilon}{E_1^2\omega} \qquad (4)$$

Considering next the 2x2 determinant formed by the second and third columns:

$$\det\begin{vmatrix} -scF_2^2 & -\lambda\upsilon \\ 1+\lambda E_1^2-c^2F_2^2 & \lambda E_1^2\omega \end{vmatrix} = \lambda\left[\left(1+\lambda E_1^2-c^2F_2^2\right)\upsilon - scE_1^2F_2^2\omega\right]=0.$$

Since we ignore the root $\lambda=0$, this equation leads us to consider the following cases:

- $\upsilon=0$ with one of the following:
  - $s=0$
  - $c=0$
  - $\omega=0$
- $\lambda = \dfrac{\left(c^2F_2^2-1\right)\upsilon + scE_1^2F_2^2\omega}{E_1^2\upsilon}$ (5)

We need both of these 2x2 determinants to vanish, so essentially we must consider the intersection of these constraints. We will first briefly show that $s=0$ leads to no new cases. Then we examine:

- $c=0$
- $\upsilon=\omega=0$
- $\lambda = \dfrac{\left(1-s^2F_2^2\right)E_1^2\omega - scF_2^2\upsilon}{E_1^2\omega} = \dfrac{\left(c^2F_2^2-1\right)\upsilon + scE_1^2F_2^2\omega}{E_1^2\upsilon}.$

Case 2a: $\lambda\neq1$, s=0 (hence c=1)

The axis of the second cone is $(0,s,c)=(0,0,1)$; thus case 2a deals with two cones whose axes are parallel. The pencil is therefore:

$$Q(\lambda)=\begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & 1-\lambda & 0 & -\upsilon \\ 0 & 0 & 1+\lambda E_1^2-F_2^2 & E_2^2\omega \\ -\mu & -\upsilon & E_2^2\omega & \mu^2+\upsilon^2+\omega^2-F_2^2\omega^2 \end{pmatrix}$$

Now Cofactor $Q_{44}=0$ implies that $1+\lambda E_1^2-F_2^2=0$. Furthermore forcing Cofactor Q_{11} = Cofactor Q_{33} = 0, we come to the following conclusions:

$$\mathit{Cofactor}Q_{11} = \det\begin{vmatrix} 1-\lambda & 0 & -\upsilon \\ 0 & 0 & E_2^2\omega \\ -\upsilon & E_2^2\omega & \mu^2+\upsilon^2+\omega^2-F_2^2\omega^2 \end{vmatrix} = (1-\lambda)E_2^4\omega^2=0.$$

$$\therefore \omega=0.$$

$$Cofactor Q_{33} = \det\begin{vmatrix} 1-\lambda & 0 & -\mu \\ 0 & 1-\lambda & -\upsilon \\ -\mu & -\upsilon & \mu^2+\upsilon^2 \end{vmatrix}$$

$$= (1-\lambda)^2(\mu^2+\upsilon^2) - (1-\lambda)(\mu^2+\upsilon^2) = -\lambda(1-\lambda)(\mu^2+\upsilon^2) = 0.$$

$$\therefore \mu = \upsilon = 0.$$

Therefore the cone vertices, and hence too the cone axes since they are parallel, must coincide. But we have already considered this configuration in case 1a(ii).

Case 2b: λ≠1, c=0 (hence s=1)

The axis of the second cone is (0,*s*,*c*)=(0,1,0); thus case 2b deals with two cones whose axes are perpendicular. The pencil is therefore:

$$Q(\lambda) = \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & -E_2^2-\lambda & 0 & E_2^2\upsilon \\ 0 & 0 & \lambda E_1^2+1 & -\omega \\ -\mu & E_2^2\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-F_2^2\upsilon^2 \end{pmatrix}$$

Forcing Cofactor Q_{44} to be zero gives us:

$$Cofactor Q_{44} = \det\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -E_2^2-\lambda & 0 \\ 0 & 0 & \lambda E_1^2+1 \end{vmatrix} = -(1-\lambda)(E_2^2+\lambda)(\lambda E_1^2+1).$$

Since λ≠1 by assumption, we have two subcases to consider:

- $\lambda = -\frac{1}{E_1^2}$
- $\lambda = -E_2^2$

Case 2b(i): $\lambda = -\frac{1}{E_1^2} \neq -E_2^2$, (λ≠1), c=0 (hence s=1)

In this case, the pencil becomes:

$$Q\left(-\frac{1}{E_1^2}\right) = \begin{pmatrix} F_1^2/E_1^2 & 0 & 0 & -\mu \\ 0 & -E_2^2+1/E_1^2 & 0 & E_2^2\upsilon \\ 0 & 0 & 0 & -\omega \\ -\mu & E_2^2\upsilon & -\omega & \mu^2+\upsilon^2+\omega^2-F_2^2\upsilon^2 \end{pmatrix} \quad (6)$$

Forcing various cofactors to be zero gives us:

$$CofactorQ_{22} = \det\begin{vmatrix} F_1^2/E_1^2 & 0 & -\mu \\ 0 & 0 & -\omega \\ -\mu & -\omega & \mu^2+\upsilon^2+\omega^2-F_2^2\upsilon^2 \end{vmatrix} = -\left(\frac{F_1^2}{E_1^2}\right)\omega^2 = 0.$$

$$\therefore \omega = 0.$$

$$CofactorQ_{33} = \det\begin{vmatrix} F_1^2/E_1^2 & 0 & -\mu \\ 0 & -E_2^2+1/E_1^2 & E_2^2\upsilon \\ -\mu & E_2^2\upsilon & \mu^2-E_2^2\upsilon^2 \end{vmatrix} = -\frac{1}{E_1^2}\left[\left(1+\frac{1}{E_1^2}\right)E_2^2\upsilon^2 - \left(\frac{1}{E_1^2}-E_2^2\right)\mu^2\right] = 0.$$

$$\therefore \left(E_1^2+1\right)E_2^2\upsilon^2 = \left(1-E_1^2E_2^2\right)\mu^2.$$

$$\therefore \upsilon = \pm\sqrt{1-\tan^2\alpha_1\tan^2\alpha_2}\cos\alpha_1\cot\alpha_2\mu.$$

Since the axes are perpendicular and $\omega=0$, the axis line of the second cone lies in the plane determined by the vertex and axis of the first cone. The geometric significance of μ and υ can therefore be interpreted as follows: μ is the distance from the vertex of the first cone to the axis line of the second; υ is the distance from the vertex of the first cone to the plane determined by the vertex and axis of the second cone; and $\sqrt{(\mu^2+\upsilon^2)}$ is the distance between the two vertices.

When the conditions of this case are satisfied, the resulting intersection is a two point tangency. This follows from the fact that the surface in Equation (6) is a straight line and hence it intersects each cone in at most two points. Levin [6] shows that when the rank of both Q and Cofactor Q_{44} is two (as it is in (6)), the surface is a line (a pair of intersecting imaginary planes) if the quantity

$$T_2 = Q_{11}Q_{22} + Q_{11}Q_{33} + Q_{22}Q_{33} - Q_{12}^2 - Q_{23}^2 - Q_{13}^2 \geq 0.$$

Substituting values from (6):

$$T_2 = \left(\frac{F_1^2}{E_1^2}\right)\left(-E_2^2+\frac{1}{E_1^2}\right) + 0 + 0 - 0 - 0 - 0 = \left(\frac{F_1^2}{E_1^2}\right)\left(-E_2^2+\frac{1}{E_1^2}\right)$$

This quantity is positive if and only if the final parenthesized term is positive. Our constraint on υ above tells us that the quantity $E_1^2E_2^2$ must be less than 1 in order for the intersection to be real. From this it follows that the final parenthesized term is positive; hence the surface defined in (6) is a line. Figure 7 illustrates two cones satisfying the relationships of this case and shows their two points of tangency.

Case 2b(ii): $\lambda = -E_2^2 \neq -\frac{1}{E_1^2}$, **($\lambda \neq 1$), c=0 (hence s=1)**

In this case we consider constraints imposed by forcing the following cofactors to be zero:

$$CofactorQ_{33} = \det \begin{vmatrix} 1-\lambda & 0 & -\lambda \\ 0 & 0 & E_2^2\upsilon \\ -\mu & E_2^2\upsilon & \mu^2+\upsilon^2+\omega^2-F_2^2\upsilon^2 \end{vmatrix} = -(1-\lambda)E_2^4\upsilon^2.$$

$$\therefore \upsilon = 0.$$

$$CofactorQ_{22} = \det \begin{vmatrix} 1-\lambda & 0 & -\mu \\ 0 & \lambda E_1^2+1 & -\omega \\ -\mu & -\omega & \mu^2+\omega^2 \end{vmatrix} = (1-\lambda)\left[(\lambda E_1^2+1)(\mu^2+\omega^2)-\omega^2\right]-\mu^2(\lambda E_1^2+1)=0.$$

$$\therefore \omega^2 = \frac{(1-E_1^2E_2^2)}{E_1^2F_2^2}\mu^2$$

$$\therefore \omega = \pm\sqrt{1-\tan^2\alpha_1\tan^2\alpha_2}\cot\alpha_1\cos\alpha_2\mu.$$

This turns out to be the same as the case just analyzed with the roles of the cones reversed. That is, the geometric significance of μ and ω is: μ is the distance from the vertex of the second cone to the axis line of the first; ω is the distance from the vertex of the second cone to the plane determined by the vertex and axis of the first cone; and $\sqrt{(\mu^2+\omega^2)}$ is the distance between the two vertices. An argument analogous to that above also leads to the conclusion that the surface in the pencil which results from using the current value of λ is a line, and the resulting intersection is a two-point tangency.

Case 2b(iii): $\lambda = -E_2^2 = -\frac{1}{E_1^2}$, **(λ≠1), c=0 (hence s=1)**

From cases 2b(i) and 2b(ii), we know that υ=ω=0. Therefore, $Q(\lambda)$ degenerates to the following equation:

$$(1+E_2^2)x^2 - 2\mu x + \mu^2 = 0$$
$$\Rightarrow (x-\mu)^2 + E_2^2x^2 = 0.$$

The only real solutions to this equation require x=μ and x=0, from which we conclude that μ=0.This tells us that the vertices must coincide, but we have already treated this possibility in case 1b.

Case 2c: λ≠1, υ=ω=0

Under the assumption that υ=ω=0, the pencil becomes:

$$Q(\lambda) = \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & 1-\lambda-s^2F_2^2 & -scF_2^2 & 0 \\ 0 & -scF_2^2 & 1+\lambda E_1^2-c^2F_2^2 & 0 \\ -\mu & 0 & 0 & \mu^2 \end{pmatrix}$$

Forcing Cofactor Q_{23} to vanish:

$$Cofactor Q_{23} = \det\begin{vmatrix} 1-\lambda & 0 & -\mu \\ 0 & -scF_2^2 & 0 \\ -\mu & 0 & \mu^2 \end{vmatrix} = \lambda scF_2^2\mu^2.$$

This leads to three possibilities, all of which have been analyzed by now:

- $s=0$ parallel axes (case 2a)
- $c=0$ perpendicular axes (case 2b)
- $\mu=0$ coincident vertices (case 1b)

Case 2d: $\lambda = \dfrac{(1-s^2F_2^2)E_1^2\omega - scF_2^2\upsilon}{E_1^2\omega} = \dfrac{(c^2F_2^2-1)\upsilon + scE_1^2F_2^2\omega}{E_1^2\upsilon}$

We shall derive a pair of constraints for this case. The first arises from equating the two indicated values of λ. This will force the second and third rows (and the second and third columns) of $Q(\lambda)$ to be linearly dependent, thereby guaranteeing the rank of $Q(\lambda)$ will be no greater than 3. We, of course, require it to be no greater than 2, and we shall establish the necessary conditions for that after deriving the first constraint.

Equating the two values for λ gives us the first constraint:

$$scF_2^2\upsilon^2 + \left[F_2^2\left(c^2+s^2E_1^2\right) - F_1^2\right]\upsilon\omega + scE_1^2F_2^2\omega^2 = 0. \tag{7}$$

In order to establish the second constraint, we first perform some elementary row and column operations on $Q(\lambda)$ in order to simplify the algebra. Since in this case the second and third rows are linearly dependent, we can multiply the second by υ, the third by ω, and add the result to the fourth row without changing any of the 3x3 subdeterminants of $Q(\lambda)$. Similarly, since the second and third columns are linearly dependent, we can multiply the second by υ, the third by ω, and add the result to the fourth column. These operations leave us with the following matrix:

$$Q(\lambda) = \begin{pmatrix} 1-\lambda & 0 & 0 & -\mu \\ 0 & 1-\lambda-s^2F_2^2 & -scF_2^2 & -\lambda\upsilon \\ 0 & -scF_2^2 & 1+\lambda E_1^2 - c^2F_2^2 & \lambda E_1^2\omega \\ -\mu & -\lambda\upsilon & \lambda E_1^2\omega & \mu^2 - \lambda\upsilon^2 + \lambda E_1^2\omega^2 \end{pmatrix} \tag{8}$$

We then force Cofactor Q_{23} to vanish. Multiplying the first row of this 3x3 subdeterminant by μ and adding it to the third row, and then multiplying the first column by μ and adding it to the third column does not alter the determinant, and we get:

$$Cofactor Q_{23} = \det\begin{vmatrix} 1-\lambda & 0 & -\lambda\mu \\ 0 & -scF_2^2 & \lambda E_1^2\omega \\ -\lambda\mu & -\lambda\upsilon & \lambda\left(E_1^2\omega^2 - \mu^2 - \upsilon^2\right) \end{vmatrix} = 0.$$

Since $\lambda \neq 0$, we can remove the common factor of λ from the third column and then expand the determinant to find:

$$(1-\lambda)\left[scF_2^2\left(\upsilon^2-E_1^2\omega^2\right)+\lambda E_1^2\upsilon\omega\right]+scF_2^2\mu^2=0.$$

Substituting for λ from (4) and manipulating, we find:

$$sF_2^2\left(sE_1^2\omega+c\upsilon\right)\left[scF_2^2\left(\upsilon^2-E_1^2\omega^2\right)+\left(1-s^2F_2^2\right)E_1^2\upsilon\omega-scF_2^2\upsilon^2\right]+scE_1^2F_2^2\mu^2\omega=0.$$

Since we have already considered s=0, we can divide out the common factor sF_2^2. Doing so and simplifying the term in brackets we obtain:

$$E_1^2\omega\left(sE_1^2\omega+c\upsilon\right)\left[\left(1-s^2F_2^2\right)\upsilon-scF_2^2\omega\right]+cE_1^2\mu^2\omega=0.$$

Now we can divide through by $E_1^2\omega$. Doing so and expanding we find:

$$\left(1-s^2F_2^2\right)c\upsilon^2+\left[\left(1-s^2F_2^2\right)sE_1^2-sc^2F_2^2\right]\upsilon\omega-s^2cE_1^2F_2^2\omega^2+c\mu^2=0.$$

Now we get lucky. Regrouping the terms in this equation, we obtain:

$$-s\left\{scF_2^2\upsilon^2+\left[\left(c^2+s^2E_1^2\right)F_2^2-E_1^2\right]\upsilon\omega+scE_1^2F_2^2\omega^2\right\}+\left(c\upsilon^2+c\mu^2\right)=0.$$

But the term in braces is nearly the same as our first constraint (7)! The only difference is that we have a $-E_1^2$ here instead of the $-F_1^2$ in (7). Since $F_1^2=E_1^2+1$, we can subtract $\upsilon\omega$ from the term inside the braces (hence adding $s\upsilon\omega$ to the equation) and balance this by subtracting $s\upsilon\omega$ from the final parenthesized term. This makes the term in the braces identical to constraint (7); hence it is zero. The second constraint therefore simplifies to:

$$c\mu^2+c\upsilon^2-s\upsilon\omega=0. \tag{9}$$

In order to use the pair of constraints expressed in (7) and (9), we need to describe μ, υ, and ω in terms of geometric invariants. (Note that s=sinθ and c=cosθ, where θ=angle between the axes; that is, s and c already have geometrically invariant characterizations.) Let V_1 and $\mathbf{a}_1$ be the vertex and unit axis vector, respectively, of the first cone, but in general position. Define V_2 and $\mathbf{a}_2$ similarly for the second cone. Recall that the vertex of the second cone in relative canonical position is (μ,υ,ω) and its axis is (0,s,c). We now reinterpret μ, υ, and ω in terms of the V_i and $\mathbf{a}_i$ in arbitrary position. Consider the planes: P_μ defined by V_1 and $\mathbf{n}=\mathbf{a}_1\mathrm{x}\mathbf{a}_2$; P_υ defined by V_1 and $\mathbf{a}_1\mathrm{x}\mathbf{n}$; and P_ω defined by V_1 and $\mathbf{a}_1$. The geometric interpretations of μ, υ, and ω are then given by:

- μ = distance between V_2 and P_μ
- υ = distance between V_2 and P_υ
- ω = distance between V_2 and P_ω

Understanding this pair of constraints in invariant terms is of somewhat limited value since the constraints are rather complex. As we will now demonstrate, however, the need to check for these constraints in a real implementation is questionable at best. We pursue two subcases: (i) μ=0 which implies that the axes intersect, and (ii) μ≠0 which implies the axes are skew.

Case 2d(i): $\lambda \neq 1$, $\mu = 0$ (axes intersect)

We shall demonstrate that it is impossible for this subcase to occur. Recall the expression for $Q(\lambda)$ in (8). In order for this matrix to have rank less than or equal to 2, it is necessary for Cofactor Q_{11} to have rank less than or equal to 1. This means that all of its 2x2 subdeterminants must vanish. We followed this line of reasoning earlier to arrive at a pair of expressions for λ (i.e., equations (4) and (5) which we repeat here):

- $$\lambda = \frac{\left(1 - s^2 F_2^2\right) E_1^2 \omega - scF_2^2 \upsilon}{E_1^2 \omega} \qquad (10)$$
- $$\lambda = \frac{\left(c^2 F_2^2 - 1\right)\upsilon + scE_1^2 F_2^2 \omega}{E_1^2 \upsilon} \qquad (11)$$

We now consider one additional 2x2 subdeterminant to arrive at a third expression for λ.

$$\mathit{Cofactor} Q_{23} = \det\begin{vmatrix} -scF_2^2 & \lambda E_1^2 \omega \\ -\lambda \upsilon & \lambda E_1^2 \omega^2 - \lambda \upsilon^2 \end{vmatrix} = \lambda scF_2^2\left(\upsilon^2 - E_1^2 \omega^2\right) + \lambda^2 E_1^2 \upsilon \omega = 0$$

$$\therefore \lambda = \frac{scF_2^2\left(E_1^2 \omega^2 - \upsilon^2\right)}{E_1^2 \upsilon \omega}. \qquad (12)$$

We now have three expressions for λ which must be satisfied simultaneously. Equating (10) and (12) and solving for ω in terms of υ, we get:

$$\omega = \frac{\left(1 - s^2 F_2^2\right)}{scF_2^2}\upsilon.$$

Equating (11) and (12) and solving for ω in terms of υ, we get:

$$\omega = \frac{scF_2^2}{\left(1 - c^2 F_2^2\right)}\upsilon.$$

These two expressions for ω must be satisfied simultaneously; hence we equate them and manipulate to discover:

$$\left(1 - s^2 F_2^2\right)\left(1 - c^2 F_2^2\right) = s^2 c^2 F_2^4$$
$$\therefore 1 = \left(s^2 + c^2\right) F_2^2 = F_2^2 = \sec^2 \alpha_2$$
$$\therefore 1 = \cos \alpha_2$$
$$\therefore \alpha_2 = 0.$$

Since we require our cones to have positive half-angle, we conclude that this case cannot occur.

Case 2d(ii): $\lambda \neq 1$, $s \neq 0$, $c \neq 0$, $\mu \neq 0$ (axes are skew)

We shall demonstrate that this subcase always leads to a two point tangency intersection. The need to check explicitly for this case depends upon the requirements of a particular implementation. In our approach, for example, we detect the degeneracy and compute the two points of tangency, but without explicitly checking the constraint equations (7) and (9) derived above [8].

The proof that this case always leads to a two-point tangency (or an intersection with no real components at all) proceeds by demonstrating that for the value of λ being considered in case 2d, $Q(\lambda)$ is always a straight line. To show this, we need the following theorem.

> *Theorem 3:* Let Q be a quadric surface, and let Q_u be the upper 3x3 submatrix of Q (i.e., det Q_u = Cofactor Q_{44}). Then Q represents a straight line if and only if both of the following conditions hold:
> 1. Rank $Q \leq 2$
> 2. The two non-zero eigenvalues of Q_u have the same sign.
>
> *Proof:* The proof follows immediately from the classification scheme for quadric surfaces given by Levin ([6], page 556).*QED*.

If the rank of Q is no greater than 2, then the rank of Q_u is also no greater than 2, and it must have a zero eigenvalue. Since Q_u is a symmetric 3x3 matrix, it has three real eigenvalues. We have already established conditions here in case 2d so that the rank Q is no greater than 2. We therefore need only show that the second condition of Theorem 3 holds in case 2d(ii). We do so by explicitly computing the eigenvalues of Q_u.

$$
\begin{aligned}
\det\left|Q_u - \chi I\right| &= \det\begin{vmatrix} 1-\lambda-\chi & 0 & 0 \\ 0 & 1-\lambda-s^2F_2^2-\chi & -scF_2^2 \\ 0 & -scF_2^2 & 1+\lambda E_1^2 - c^2F_2^2-\chi \end{vmatrix} \\
&= (1-\lambda-\chi) \\
&\quad \left\{\chi^2 - \left[\left(1-\lambda-s^2F_2^2\right)+\left(1+\lambda E_1^2-c^2F_2^2\right)\right]\chi + \det\begin{vmatrix} 1-\lambda-s^2F_2^2 & -scF_2^2 \\ -scF_2^2 & 1+\lambda E_1^2-c^2F_2^2 \end{vmatrix}\right\}
\end{aligned}
$$

Note that the determinant in the final expression is just the interior 2x2 subdeterminant of Q in (8). Since we know that the second and third rows of this matrix are linearly dependent, this determinant must be zero. The three eigenvalues of Q_u are therefore:

$$
\begin{aligned}
\chi_1 &= 1-\lambda \\
\chi_2 &= \left(1-\lambda-s^2F_2^2\right)+\left(1+\lambda E_1^2-c^2F_2^2\right) = (1-\lambda)\left(1-E_1^2\right)+\left(E_1^2-E_2^2\right) \\
\chi_3 &= 0
\end{aligned}
$$

As expected, one of them is zero. We need to show that the other two have the same sign. We substitute the value of λ from (10) into the expressions for χ_1 and χ_2 and simplify to obtain:

$$E_1^2\omega\chi_1 = s^2E_1^2F_2^2\omega + scF_2^2\upsilon \tag{13}$$

$$E_1^2\omega\chi_2 = -E_1^2\left[\left(c^2 + s^2E_1^2\right)F_2^2 - F_1^2\right]\omega + \left(1 - E_1^2\right)scF_2^2\upsilon \quad (14)$$

Note that we can assume without loss of generality that $\omega \geq 0$ and $s \geq 0$. If ω happens to be negative, we simply negate $\mathbf{a}_1$. This forces ω to be nonnegative without affecting the surface or the intersection. Similarly, if s happens to be negative, we negate $\mathbf{a}_2$. With these observations, we can complete our treatment of this case by considering two subcases: (a) $c\upsilon<0$, and (b) $c\upsilon \geq 0$.

Case 2d(ii.a): $c\upsilon<0$

Consider the constraint of Equation (9). Since c and υ have opposite signs, the coefficients of μ^2, υ^2, and ω all have the same sign. Since u^2 and v^2 are nonnegative, and we have arranged for $\omega \geq 0$, the only solution is $\mu=\upsilon=\omega=0$. This means the vertices coincide, but we have already handled this possibility in case 1b.

Case 2d(ii.b): $c\upsilon \geq 0$

Consider the constraint of Equation (7). Since $c\upsilon \geq 0$, c and υ have the same sign. Therefore if the coefficient of the $\upsilon\omega$ term is nonnegative, then all the terms in (7) would have the same sign. This would mean that the only solution would be $\upsilon=\omega=0$. Substituting this result into (9) would give us that $\mu=0$, and we would be back at the coincident vertex case. Therefore, the only way to arrive at a new case is for the coefficient of the $\upsilon\omega$ term to be negative. That is, we require:

$$F_2^2\left(c^2 + s^2E_1^2\right) - F_1^2 < 0 \quad (15)$$

By rearranging the left-hand side of this inequality we observe:

$$F_2^2\left(c^2 + s^2E_1^2\right) - F_1^2 = \left(E_1^2 - 1\right)s^2F_2^2 + \left(E_2^2 - E_1^2\right)$$

Now it is clear that if $E_2 \geq E_1 \geq 1$, then the right-hand side of this equation is positive, and hence the previous inequality is *not* satisfied. We can reverse the roles of the two quadrics by putting the second in canonical position and the first in relative canonical position. We then arrive at the symmetric result, namely that $E_1 \geq E_2 \geq 1$ will fail to produce a new case. It therefore follows that at least one of the E_i must be less than 1. Without loss of generality, we assume $E_1<1$.

In summary, then, for case 2d(ii.b) to yield a previously unexamined configuration, we must have the following three conditions:

- $c\upsilon \geq 0$
- $E_1<1$
- $F_2^2\left(c^2 + s^2E_1^2\right) - F_1^2 < 0$

From Equation (13), the first bullet above, and the fact that s and ω are nonnegative, it is clear that the sign of χ_1 is positive. From Equation (14), all three of the conditions just noted, and the fact that s and ω are nonnegative, it is clear that the sign of χ_2 is positive. Therefore by Theorem 3, the surface defined in (8) is a straight line, and case 2d can yield only a pair of tangent points or no real intersection at all.

Figure 8 illustrates two cones satisfying constraints (7) and (9) and also satisfying the three additional conditions listed here. Their two points of tangency are also shown.

5.0 Summary

We have presented an algebraic analysis, the collected results of which enumerate all possible cases in which a pair of natural quadrics have planar intersections. These results are intended to be a prescription for the design of a purely geometric computer-based implementation. Although our algebraic analysis is long and difficult, our final geometric constraints are short and simple. Thus having done this analysis here once and for all, it is easy to implement tests for the resulting geometric conditions.

Table II summarizes all the configurations in which pairs of natural quadrics intersect in planar curves. Situations giving rise to two point tangencies are summarized in Table III. The conditions are stated in terms of geometric constraints between pairs of natural quadrics described by their geometric parameters in general position and orientation. No coordinate system transformations of any sort are required to apply these tests or to compute the resulting planar curves. The specific details of the calculation of the resulting planar curves are presented in a companion paper [8].

We could extend the analysis presented here in two directions. First, we could attempt a systematic analysis of conditions giving rise to one point tangencies. In our opinion, it is not worth the added effort to do so. Not only are such conditions of questionable value in modeling, but also we know only a sufficient condition yielding one point tangencies, namely that the pencil contain a single point surface. We were forced to consider two point tangencies since the line has rank 2 when represented as a quadric surface, but the single point has rank 3 and therefore was not included as a byproduct of our analysis.

Second, we could attempt a systematic analysis of other quadric surfaces such as ellipsoids, paraboloids, and hyperboloids. Such a general extension would likely be extremely difficult and probably is not practical. Note how difficult the cone-cone case was to analyze. Still it might be worth the effort in order to verify some of the conjectures advanced below or to support select quadrics such as the ellipsoid which do occasionally arise in solid modeling applications.

To close, we wish to draw particular attention to the following configurations leading to planar intersections:

- Cylinder-Cylinder
  intersecting axes; equal radii

- Cylinder-Cone
  intersecting axes; $r=d\sin\alpha$ where d is the distance from the cone vertex to the point at which the axes intersect

- Cone-Cone
  intersecting axes; $d_1\sin\alpha_1=d_2\sin\alpha_2$ where d_i is the distance from vertex i to the point at which the axes intersect

We can summarize these three results in a single statement:

> Let Q_1 and Q_2 be two natural quadrics (cylinders and cones) whose axes intersect. The intersection is planar independent of the angle θ between the axis lines if and only if the distances from the point of intersection of the axes to each of the surfaces are the same.

We also draw attention to the following results for the natural quadrics. Planar curves of intersection can occur only if the axes are parallel, coincident, or intersecting. They never arise when the axes are skew. On the other hand, two point tangencies arise only if the axes are skew. Furthermore, planar curves arise only in conjunction with $\lambda=1$. Two point tangencies are all byproducts of the other two values of λ.

We would like to know how these statements generalize to other quadrics of revolution, to other axial quadrics which are not quadrics of revolution, and to other surfaces of revolution which are not quadric surfaces. The results of an initial attempt to generalize these observations to other quadrics of revolution -- paraboloids, ellipsoids, and hyperboloids -- are presented in [4].

Table II
Summary of Conditions Giving Rise to Planar Intersection Curves Between Pairs of Natural Quadric Surfaces

Section	Surface Pair	Case	Geometric Conditions	Results	Figure
4.1	sphere/sphere		All	empty; one tangent point; or one circle	
4.2	sphere/cylinder		Center of sphere on axis of cylinder	empty; one tangent circle; or two circles	
4.3	sphere/cone	1	Center of sphere on axis of cone	empty; one tangent circle; one circle + vertex; or two circles	
4.4	cylinder/cylinder	2	Parallel axes	empty; one tangent line; or two lines	
		3	Intersecting axes & equal radii	two ellipses	
4.5	cylinder/cone	2a	Coincident axes	two circles	
		2b	Axes intersect in a point at distance $d=r/\sin\alpha$ from the vertex of the cone	two ellipses (same or opposite halves of the cone); or one ellipse & tangent line	2
4.6	cone/cone	1a(i)	Parallel axes, same cone angle	ellipse; shared tangential ruling; or hyperbola	4
		1a(ii)	Coincident axes	two circles or single vertex	
		1b	Axes intersect at point I such that $d_1\sin\alpha_1=d_2\sin\alpha_2$ where d_i is the distance from vertex i to I. (This includes the case where the vertices coincide; i.e., $d_1=d_2=0$.)	various combinations of pairs of conics *or* a tangent line plus a conic (1-4 lines or a single point if the vertices coincide)	6

Table III
Summary of Conditions Giving Rise to Two Point Tangencies Between Pairs of Natural Quadric Surfaces

Section	Surface Pair	Case	Geometric Conditions	Figure
4.3	sphere/cone	2	Center of sphere in plane (V,**a**) at distance $d=r/\cos\alpha$ from vertex	1
4.5	cylinder/cone	3	Skew axes; distance between axes: $d = \frac{r\sin\theta}{\sqrt{\sin^2\theta - \sin^2\alpha}}$	3
4.6	cone/cone	2b(i)	Perpendicular axes; V_2 in plane of (V_1,$\mathbf{a}_1$); distance, μ, from V_1 to axis line (V_2,$\mathbf{a}_2$) and distance, υ, from V_1 to plane (V_2,$\mathbf{a}_2$) are related as: $\upsilon = \sqrt{1-\tan^2\alpha_1\tan^2\alpha_2}\cos\alpha_1\cot\alpha_2\mu$	7
		2b(ii)	Same as 2b(i) with roles of the cones reversed	7
		2d(ii)	Skew axes with constraints (7) and (9)	8

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