Projection Matrix Summary

0. Recall, we build all 4x4 matrices here as a product: \( \mathbf{M}_{ec-lds} = \mathbf{M}_{wv} \ast \mathbf{M}_{proj} \) where:

- \( \mathbf{M}_{wv} \) does the window-viewport map into the -1...+1 logical device space of OpenGL
- \( \mathbf{M}_{proj} \) does the 3D to 2D projection with preservation of (at least relative) depth.

1. **Orthogonal** *(Given: \( x_{\min}, x_{\max}, y_{\min}, y_{\max}, z_{\min}, z_{\max} \), all specified in eye coordinates with \( x_{\min} < x_{\max}, y_{\min} < y_{\max}, \) and \( z_{\min} < z_{\max} \))*

   \( \mathbf{M}_{proj} \) is the identity matrix since there is nothing that needs to be done. \( \mathbf{M}_{wv} \) simply maps \( (x_{\min}, x_{\max}, y_{\min}, y_{\max}, z_{\min}, z_{\max}) \) to \((-1, 1, -1, 1, 1, -1)\). (Note the reversal in the z direction.)

   This yields three pairs of equations with two unknowns:

   \[
   \begin{align*}
   a_x x_{\min} + b_x &= -1 \\
   a_x x_{\max} + b_x &= 1 \\
   a_y y_{\min} + b_y &= -1 \\
   a_y y_{\max} + b_y &= 1 \\
   a_z z_{\min} + b_z &= 1 \\
   a_z z_{\max} + b_z &= -1
   \end{align*}
   \]

   Solving for \( a_x, b_x, a_y, b_y, a_z, \) and \( b_z \), we get:

   \[
   \begin{align*}
   a_x &= 2/(x_{\max} - x_{\min}); \\
   b_x &= -(x_{\max} + x_{\min})/(x_{\max} - x_{\min}) \\
   a_y &= 2/(y_{\max} - y_{\min}); \\
   b_y &= -(y_{\max} + y_{\min})/(y_{\max} - y_{\min}) \\
   a_z &= -2/(z_{\max} - z_{\min}); \\
   b_z &= (z_{\max} + z_{\min})/(z_{\max} - z_{\min})
   \end{align*}
   \]

   Hence

   \[
   \mathbf{M}_{wv} = \begin{pmatrix}
   a_x & 0 & 0 & b_x \\
   0 & a_y & 0 & b_y \\
   0 & 0 & a_z & b_z \\
   0 & 0 & 0 & 1
   \end{pmatrix}
   \]

   Finally, \( \mathbf{M}_{ec-lds} = \mathbf{M}_{wv} \mathbf{M}_{proj} = \mathbf{M}_{wv} \mathbf{I} = \mathbf{M}_{wv} \).

2. **Oblique** *(Given: \( z_{pp}, x_{\min}, x_{\max}, y_{\min}, y_{\max}, z_{\min}, z_{\max} \) and \( \mathbf{d} = (d_x, d_y, d_z) \), the common direction of projection, all specified in eye coordinates with \( x_{\min} < x_{\max}, y_{\min} < y_{\max}, z_{\min} < z_{\max} \), and \( d_z \neq 0 \))*

   \( \mathbf{M}_{proj} \) can be shown to be:

   \[
   \mathbf{M}_{proj} = \begin{pmatrix}
   1 & 0 & -d_x/d_z & z_{pp}d_x/d_z \\
   0 & 1 & -d_y/d_z & z_{pp}d_y/d_z \\
   0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 1
   \end{pmatrix}
   \]

   Clearly \( \mathbf{M}_{wv} \) is the same for oblique as for orthogonal, hence:
where $a_x, b_x, a_y, b_y, a_z,$ and $b_z$ are as given in equation (1) above.

3. Perspective (Given: $z_{pp}, x_{min}, x_{max}, y_{min}, y_{max}, z_{min}, z_{max}$ all specified in eye coordinates with $x_{min}<x_{max}; y_{min}<y_{max}; z_{min}<z_{max}<0; \text{and } z_{pp}<0$)

We derive $M_{proj}$ (and, in particular, the portions of the transformation involving the eye coordinate $z$ direction) so that mapping to the $z$ range of LDS space is included in $M_{proj}$. Thus we get:

$$M_{wv} = \begin{pmatrix} a_x & 0 & 0 & b_x \\ 0 & a_y & 0 & b_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{proj} = \begin{pmatrix} z_{pp} & 0 & 0 & 0 \\ 0 & z_{pp} & 0 & 0 \\ 0 & 0 & a_z & b_z \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $a_x, b_x, a_y,$ and $b_y$ are as given in equation (1) above. The $a_z$ and $b_z$ terms can be shown to be:

$$\alpha_z = \frac{z_{min} + z_{max}}{z_{max} - z_{min}}; \quad \beta_z = \frac{2z_{min} z_{max}}{z_{max} - z_{min}}$$

Finally:

$$M_{ec-ids} = M_{wv} M_{proj} = \begin{pmatrix} a_x & 0 & 0 & b_x \\ 0 & a_y & 0 & b_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_{pp} & 0 & 0 & 0 \\ 0 & z_{pp} & 0 & 0 \\ 0 & 0 & a_z & b_z \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_x z_{pp} & 0 & b_x & 0 \\ 0 & a_y z_{pp} & b_y & 0 \\ 0 & 0 & a_z & b_z \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In principle, this matrix should be fine, but there is a clipping issue we will discuss that forces us to use the negated version of this matrix. Basically we need to be sure that the $w$ component that results when this matrix is used is positive for any points in the view frustum. Since this matrix will set $w = z$, all visible points will have negative $w$. Negating the matrix prevents that without altering how points are projected since negating all 16 elements will just produce a different (but projectively equivalent) point. Hence:

$$M_{ec-ids} = \begin{pmatrix} -a_x z_{pp} & 0 & -b_x & 0 \\ 0 & -a_y z_{pp} & -b_y & 0 \\ 0 & 0 & -a_z & -b_z \\ 0 & 0 & -1 & 0 \end{pmatrix}$$