2. Vector Analysis

2-1 Introduction

Vector analysis is the branch of mathematics that was developed to describe quantities that are both directional in nature and distributed over regions of space. The reason for starting our study of electromagnetics with vector analysis is simple; vector analysis is the language best suited for describing electromagnetic effects.

In this chapter, we will discuss the elements of vector analysis that are directly applicable to electromagnetic phenomena. Our discussion will start by defining the concept of a physical quantity, and then identifying the properties of scalar and vector fields. The remainder of our discussion will be devoted to development of the algebra and calculus of vector fields.

2-2 Physical Quantities and Units

Electromagnetics deals with phenomena and entities that can be perceived and measured. We call entities that can be measured***physical quantities*** . In physics, physical quantities are always defnined in terms of the measurment proceedure use to percieve them:

> The definition of a physical quantity is the description of the operational procedure used to measure the quantity[1].

This kind of definition is callen an *operational definition*, since it defines a physical quantity in terms of the process used to measure it.

To help us understand this definition, let us consider a common physical quantity, distance. We can define the distance between two points as the total number of measuring objects that can be laid end-to-end on a straight line between the points. The measuring object can be anything, such as a rock, a twig, or a meter stick. Since we have defined distance by telling how to measure it, any number of people can measure the distance between two points and obtain the same answer. Obviously, the accuracy to which a physical quantity can be measured depends upon how carefully one follows the measurement procedure.

Any measurement is a comparison of what is being measured and some standard. These standards are called ***units***. In the case of the distance example just presented, the unit is the object whose size is used as the measurement standard, i.e., the rock, twig, or meter stick. In order for the specification of a physical quantity to have meaning the unit must be well defined. For instance, if a rock is used as the unit, the particular rock must be clearly identified, along with how it is oriented during the measurement process.

Because they are defined in terms of how they are measured, physical quantities are always specified by one or more numbers, each with its associated unit. The number of numbers needed to specify a particular physical quantity depends on the way the quantity is defined. For instance, only one number is required to specify a distance. On the other hand, three numbers (called coordinates) are necessary to specify a position in three-dimensional space. All naturally occurring physical quantities can be represented by real numbers. However, we sometimes find it convenient to create complex-valued physical quantities from naturally occurring physical quantities. A common example is in circuit analysis, where complex phasors are used to represent sinusoidal steady state voltages and currents. In these cases, complex-valued quantities can be considered to represent two quantities - one real, and the other imaginary.

The unit of a physical quantity can be any well defined standard, but it is usually

[1]See *Elements of Physics* by Shortley and Williams, 4th edition, Prentice-Hall, 1965, pp. 4.

desirable to limit the number of units used in a measurement to as few as possible. Certain sets of quantities are, by convention, regarded as ***fundamental quantities***, specified in internationally accepted ***fundamental units***. All other units can be derived in terms of these fundamental units. By far the most accepted unit system among electrical engineers is the MKSA system[2]. The fundamental quantities in this system are *length*, *mass*, *time*, and *current*, specified in *meters*, *kilograms*, *seconds*, and *amperes*, respectively. The MKSA system is a subset of the ***International System of Units*** (SI), which also includes the *candela* (a unit of *luminous intensity*) and the *degree Kelvin* (a unit of *temperature*).

Derived units are units that are specified in terms of the fundamental units. The newton, for example, is a derived unit of force, defined as 1.0 [kg•m/sec^2]. The most common derived units used in electromagnetic analysis are summarized in Appendix A.

> The dimensions of a physical quantity are specified by the powers of the fundamental physical quantities that occur in its definition.

For example, since the unit of speed is a length unit divided by a time unit, its dimensions are (length)/(time). Similarly, the dimensions of the newton are (mass)x(length)/(time)^2. The dimensions of physical quantities are important, because two physical quantities can be added or subtracted if, and only if, they have the same dimensions. Thus, apples can be equated with apples, but not oranges. Any equation in which the units of the left and right hand sides do not agree is simply wrong.

2-2.1 Discrete and Field Quantities

The physical quantities used in electromagnetics can be either discrete or field quantities. The simplest are discrete quantities.

> ***Discrete quantities*** are defined over regions of space or at single points, but not on a point-by-point basis throughout a region.

The definition of the average temperature of a room has nothing to do with the position of the observer, so it is a discrete quantity. Similarly, the distance from Kansas City to New York is a quantity that is independent of the position of an observer.

Most of the physical quantities encountered in electromagnetics are field quantities.

> ***Field quantities*** are defined on a point-by-point basis throughout a region of space.

The temperature in a room is a field quantity, since it is defined uniquely for each point in the room. Another field quantity is wind speed, which is also a function of the position at which it is measured.

The distinction between discrete and field quantities is important, because the procedures used to describe them are different. One needs only ordinary algebra to balance a

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[2]For a complete discussion of the unit systems used in electromagnetics see *Handbook of Chemistry and Physics*, The Chemical Rubber Co., Baca Raton, Fla., 1991

checkbook or to compare the weights of two objects, since these operations involve only discrete quantities. Analyzing the characteristics of the temperature distribution inside a room is more difficult. This is because the temperature is a continuous function of position, and requires vector analysis to describe it fully.

2-2.2 Scalars and Vectors

We have just seen that physical quantities can be classified as either discrete or field quantities. They can also be classified according to the number of digits needed to specify them. The simplest are scalars.

> A ***Scalar*** is a quantity that can be specified by a single number and its associated unit.

Temperature, altitude, and weight are all scalar quantities, since each can be specified by a single number. Throughout this text, scalar quantities are represented as non-boldface symbols in italics, such as D and ρ. Also, the units of all scalars will be written in brackets, such as [kg/m].

There are also physical quantities that have both a size and a direction associated with them. These are called vectors and are defined as follows.

> A ***Vector*** is a quantity that can be specified by a ***magnitude*** and a ***direction***.

Examples of vector quantities are velocity and force. Throughout this text, boldface alphabetic characters, such as **A**, **E**, and **h**, are used to denote vector quantities[3].

The magnitude and direction of a vector are very different entities. The magnitude is a positive-valued scalar, which includes its associated unit. We will represent the magnitude of a vector **A** as either |**A**| or A. The direction, as its name implies, is a spatial orientation. For instance, the wind velocity at a point may be specified as 2.5 [m/s] in the south east direction.

By convention, any vector can be represented graphically by a line extending from a tail to a head. An arrow is placed at the head and points in the direction of the vector. The distance from the tail to the head represents the vector's magnitude. The graphical representation of a vector **A** is shown in Figure 2-1.

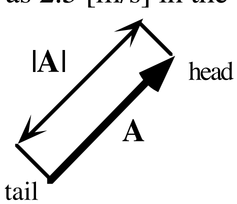

Figure 2-1

A graphical representation of a vector.

Discrete vectors are associated with regions of space, but not a specific point. An important example is the directed distance between two points.

> The ***directed distance*** $\mathbf{R}_{ab}$ between the points a and b is a vector whose magnitude equals the distance between these points and whose direction is parallel to the line directed from a to b.

Even though this definition of $\mathbf{R}_{ab}$ involves the points a and b, this vector is not defined to exist

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[3]In handwritten work, vectors are typically written as $\bar{A}$ or $\vec{A}$, since boldface characters are difficult to draw by hand.

at any particular point. As a result, its representation can be translated freely to any point, as long as its magnitude and direction are not changed. This is depicted in Figure 2-2.

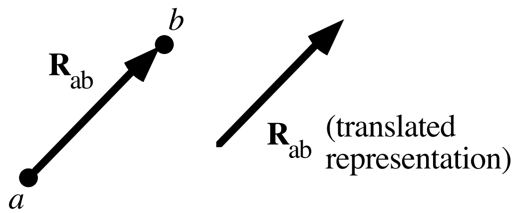


Figure 2-2

A discrete vector, shown at two different locations in space.

Figure 2-3a shows the graphical representation of a ***vector field***. Here, the vector is represented at equally spaced points in a ***quiver plot*** (sometimes called a ***needle plot***). This type of diagram is helpful in that it conveys both the magnitude and direction of the vector at a number of points. On the other hand, this type of diagram does not easily convey the sense of the vector's "flow". By flow, we mean the path that a particle would take if it were pushed by the vector (assuming that the vector represented a force field). This flow is best represented by the streamline plot shown in Figure 2-3b. Here, continuous lines called ***streamlines*** are drawn that are tangent to the vector's direction at each point. These streamlines are the paths that the vector would "push" a particle. Magnitude information is not directly conveyed by the streamlines. Nevertheless, one can usually infer this information by measuring the spacings between the streamlines, since vector magnitudes are usually strongest when the streamlines are the closest. This can be seen by comparing Figures 2-3a&b.

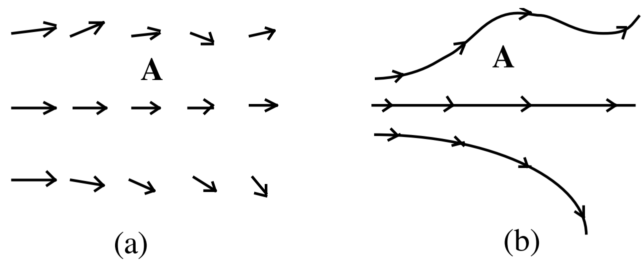


Figure 2.3

Graphical representations of a vector field: a) quiver plot, b) streamlines.

2-3 Vector Algebra

Having defined scalars and vectors, we will now define several operations involving them. These operations are essential, for without them we would have no way to formulate the mathematical equations that describe the physical processes found in electromagnetics. Three classes of operations are possible in vector algebra: scalar-scalar, scalar-vector, and vector-vector. Since the operations from the first class are already known from ordinary algebra, our discussion will be limited to operations involving vectors.

2-3.1 Addition and Subtraction of Vectors

The sum of any two vectors, **A** and **B**, is itself a vector, defined by the graphical process depicted in Figure 2-4a. Here, the vector sum **A+B** is defined as the vector that completes the parallelogram formed by **A** and **B**.

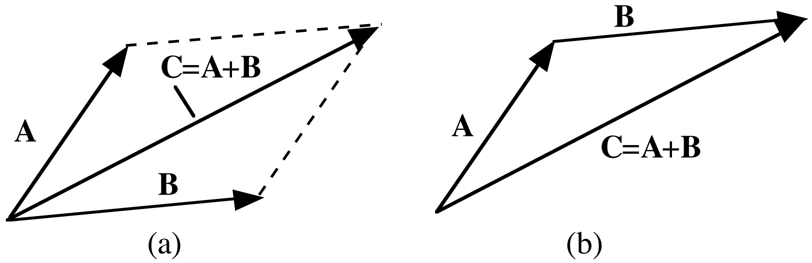


Figure 2-4
Vector Addition:
a) Completing the parallelogram, b) the head-to-tail rule.

An equivalent definition of the sum **A+B**, called the ***addition tail-to-head rule***, is depicted in Figure 2-4b, where the representation of **B** has been translated so that its tail lies at the head of A. The sum **A+B** is then defined as the vector whose representation extends from the tail of **A** to the head of the translated **B**. This definition is, in some ways, more visually descriptive than the first, but it can also be somewhat misleading for field vectors, since it implies that the representations of a vector field can be moved to any point in space as if they were discrete vectors. To counter this illusion, one must remember that this sliding process is only a tool used to define **A+B**. In reality, **A**, **B**, and **A+B** are all defined at *exactly* the same point.

Vector addition satisfies the associative and commutative laws,

Associative law: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ 2.1a

Commutative law: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. 2.1b

Both proofs are straightforward from the definition of vector addition and are left as an exercise for the reader.

Vector subtraction is defined in terms of vector addition by

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (-\mathbf{B}) \quad , \qquad 2.2$$

where the symbol "≡" means "equal by definition." The vector -**B** is called the ***negative*** of **B**; it has the same magnitude as **B**, but opposite direction. Figure 2-5a shows the graphical representation of **C=A+(-B)**. Figure 2-5b shows that **C=A-B** can also be represented using the ***subtraction tail-to-head rule***, where the representation of **A** - **B** extends from the tip of **B** to the tip of **A**. When using this rule for vector fields, however, it must be remembered that **A**, **B**, and **C** all exist at the same point, even though **C** has been translated by the graphical procedure.

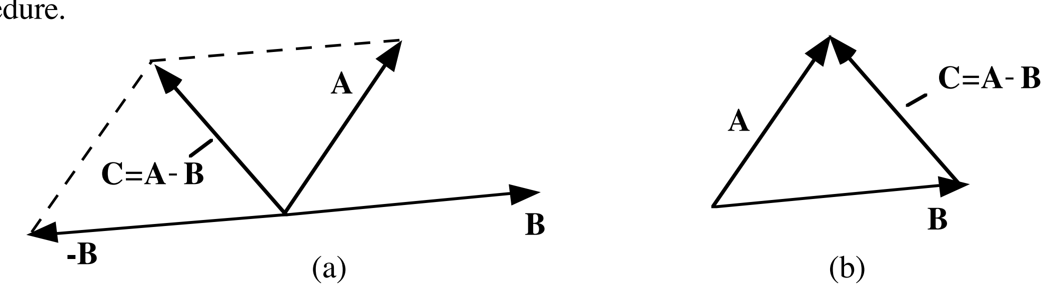


Figure 2-5: Vector Subtraction: a) Completing the parallelogram, b) head-to-tail rule

2-3.2 Multiplication of a Vector by a Scalar

The product $a\mathbf{B}$ is defined as a vector with the same direction as **B**, and magnitude equal to $|a||\mathbf{B}|$. If the sign of a is negative, the direction of the vector $a\mathbf{B}$ is opposite to that of **B**. Figure 2-6 depicts a scalar product $a\mathbf{B}$.

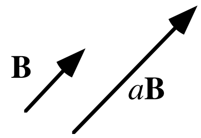


Figure 2-6:
Multiplication of a scalar and a vector

The product of a scalar and a vector obeys both the commutative and distributive laws,

Commutative law: $a\mathbf{B} = \mathbf{B}a$ 2.3

Distributive law: $a(\mathbf{B} + \mathbf{C}) = a\mathbf{B} + a\mathbf{C}$. 2.4

The quotient $\frac{\mathbf{B}}{a}$ can be defined in terms of the scalar-vector product

$$\frac{\mathbf{B}}{a} \equiv a^{-1}\mathbf{B} = \frac{1}{a}\mathbf{B} . \qquad 2.5$$

We can use the scalar-vector product to represent an arbitrary vector **A** in the form

$$\mathbf{A} = |\mathbf{A}|\,\hat{\mathbf{a}}_A = A\,\hat{\mathbf{a}}_A \quad , \qquad 2.6$$

where $|\mathbf{A}|$ and A are the magnitude of **A**, and $\hat{\mathbf{a}}_A$ is a ***unit vector*** that has the same direction as **A** and a magnitude of unity (i.e., 1.0). Multiplying both sides of equation 2.6 by $|\mathbf{A}|^{-1}$, we obtain the following expression for the unit vector $\hat{\mathbf{a}}_A$,

$$\hat{\mathbf{a}}_A = \frac{\mathbf{A}}{|\mathbf{A}|} \quad . \qquad 2.7$$

As its name implies, a unit vector has unit magnitude.

2-3.3 The Scalar (or Dot) Product of Two Vectors

There are two multiplication operators that involve two vectors. The first is called the ***scalar product***, because it produces a scalar. The scalar product of two vectors is defined as a scalar whose value is given by

$$\mathbf{A}\bullet\mathbf{B} \equiv |\mathbf{A}||\mathbf{B}|\cos\theta_{AB} \quad . \qquad 2.8$$

Here, the angle θ_{AB} is defined as the smaller angle between **A** and **B** (i.e. , $\theta_{AB} \le 180°$), and $|\mathbf{A}|$ and $|\mathbf{B}|$ are the magnitudes of **A** and **B**, respectively. The expression $\mathbf{A}\bullet\mathbf{B}$ is read as "**A** dot **B**", and the terms "scalar product" and "dot product" are used interchangeably.

When we take the dot product of a vector with itself, we obtain

$$\mathbf{A}\bullet\mathbf{A} = |\mathbf{A}||\mathbf{A}|\cos\theta_{AA} = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2 \ . \qquad 2.9$$

Thus, the magnitude of any vector can be written in terms of its dot product,

$$|\mathbf{A}| = \sqrt{\mathbf{A}\bullet\mathbf{A}} \ . \qquad 2.10$$

The dot product satisfies the commutative and distributive laws,

Commutative Law $\quad \mathbf{A}\bullet\mathbf{B} = \mathbf{B}\bullet\mathbf{A} \qquad 2.11$

Distributive Law $\quad \mathbf{A}\bullet(\mathbf{B}+\mathbf{C}) = \mathbf{A}\bullet\mathbf{B} + \mathbf{A}\bullet\mathbf{C} \ . \qquad 2.12$

The commutative law follows directly from the symmetry of the dot product. The proof of the distributive law is straightforward and is left as an exercise.

The definitions of perpendicular and collinear vectors are derived from the dot product.

> Two vectors, **A** and **B**, are ***perpendicular*** (or ***orthogonal***) if $\mathbf{A}\bullet\mathbf{B}=0$. Vectors are ***collinear*** if $|\mathbf{A}\bullet\mathbf{B}|=|\mathbf{A}||\mathbf{B}|$. Collinear vectors are ***parallel*** if $\mathbf{A}\bullet\mathbf{B}=|\mathbf{A}||\mathbf{B}|$ and are ***antiparallel*** if $\mathbf{A}\bullet\mathbf{B} = -|\mathbf{A}||\mathbf{B}|$.

The dot product is a convenient tool for finding the component of a vector along an particular direction. Figure 2-7 shows a vector **A** and reference line that is parallel to the direction of the unit vector $\hat{\mathbf{a}}_B$. Also shown is the right triangle formed by the reference line, the vector **A**, and the line that extends from the tip of **A** and intersects the reference line at a right angle. When $\theta_{AB}\leq 90°$, the component A_B of the vector **A** along the direction $\hat{\mathbf{a}}_B$ is defined as the length of the side of the right triangle that lies along the reference line. If $\theta_{AB}>90°$, then the component A_B is the negative of this distance. From this definition and Figure 2-7, it follows that the component A_B of **A** in the direction $\hat{\mathbf{a}}_B$ is given by

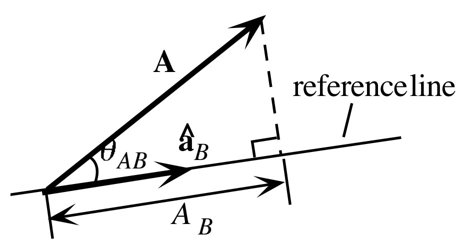


Figure 2-7: The projection of a vector along a reference line.

$$A_B = |\mathbf{A}|\cos\theta_{AB} \ .$$

But from equation 2.8, we find that $|\mathbf{A}|\cos\theta_{AB}$ can be written as the dot product $\mathbf{A}\bullet\hat{\mathbf{a}}_B$. Hence, we can write

$$A_B = \mathbf{A}\bullet\hat{\mathbf{a}}_B = |\mathbf{A}|\cos\theta_{AB} \ . \qquad 2.13$$

The dot product can be used to expand any vector as the sum of perpendicular component vectors. Consider the vector **A**, shown in Figure 2-8, which exists in 3-dimensional space.

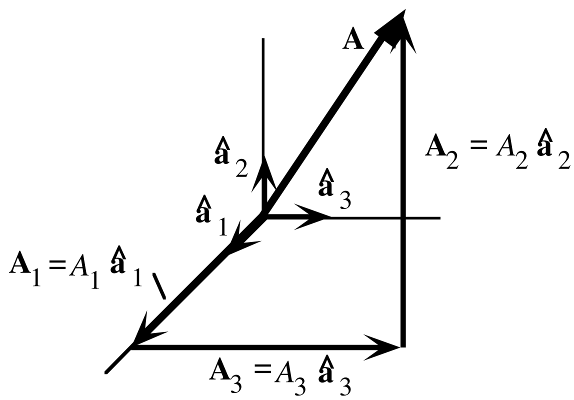


Figure 2-8
An arbitrary vector **A**, shown as the sum of three, mutually-orthogonal component vectors.

If the unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$ are mutually orthogonal (perpendicular), we can express **A** as

$$\mathbf{A} = A_1\,\hat{\mathbf{a}}_1 + A_2\,\hat{\mathbf{a}}_2 + A_3\,\hat{\mathbf{a}}_3 \ . \qquad 2.14$$

The scalars A_1, A_2, and A_3 are the components of **A** along the directions $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$, respectively. Remembering that the dot-products of perpendicular vectors are zero, we can find the components A_1, A_2, and A_3 simply by taking the dot products of **A** with $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, and $\hat{\mathbf{a}}_3$, respectively, obtaining

$$A_i = \mathbf{A}\bullet\hat{\mathbf{a}}_i \ , \quad i = 1, 2, \text{or } 3 \ . \qquad 2.15$$

2-3.4 The Vector (or Cross) Product of Two Vectors

The second product of vectors is the ***vector*** (or ***cross***) ***product***. Unlike the scalar product, which produces a scalar from two vectors, the vector product of two vectors produces another vector, defined by

$$\mathbf{A} \times \mathbf{B} \equiv \hat{\mathbf{a}}_n |\mathbf{A}||\mathbf{B}|\sin(\theta_{AB}) \ , \qquad 2.16$$

where θ_{AB} is defined as the smallest angle between **A** and **B**. The expression **AxB** is read as "**A** cross **B**", and the terms "vector product" and "cross product" are used interchangeably. Figure 2-9 shows the relationship between **A**, **B**, and **A x B**. The unit vector $\hat{\mathbf{a}}_n$ is specified by a convention called the ***right-hand rule***. This rule states that $\hat{\mathbf{a}}_n$ is perpendicular to both **A** and **B** and points in the direction of a right hand thumb when the other fingers point along the arc that **A** would follow if it were rotated into **B** through the smallest angle between them.

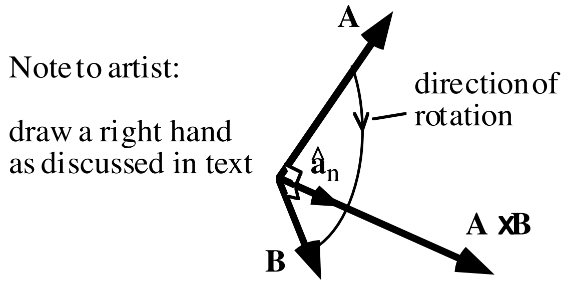


Figure 2-9:
The cross product of two vectors

The cross product obeys the distributive law,

Distributive law: **A**x(**B**+**C**) = **A**x**B** + **A**x**C** . 2.17

This can be proved directly from the definition of the cross product. On the other hand, the cross product obeys neither the commutative nor the associative laws, which can be seen from the following inequalities,

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \neq \mathbf{B}\times\mathbf{A} \quad . \tag{2.18}$$

and

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C} \quad . \tag{2.19}$$

Equation 2.18 is a direct result of the right-hand rule. Equation 2.19 is most easily proved by observing that the vector **A**x(**B**x**C**) lies in the plane formed by **B** and **C** (since **B**x**C** lies perpendicular to that plane), whereas the vector (**A**x**B**)x**C** lies in the plane containing **A** and **B**.

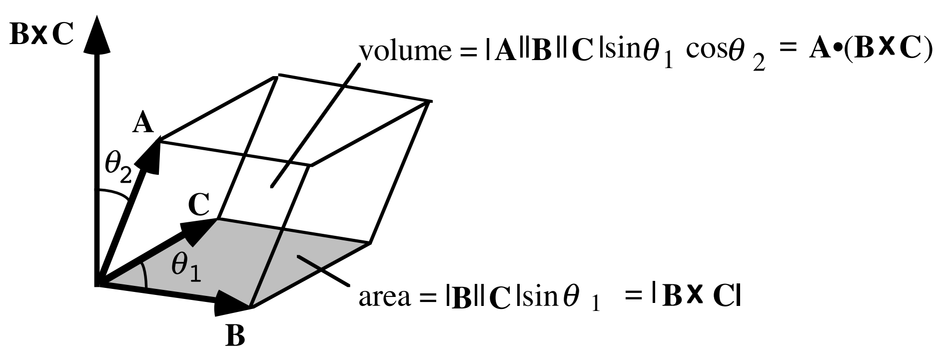
2-3.5 Products of Three Vectors

There are two combinations of products that involve three vectors. These are the ***scalar triple product*** and the ***vector triple product***, so named because they produce a scalar and a vector, respectively. The simplest of the two is the scalar triple product. For three vectors, **A**, **B**, and **C**, the scalar triple product **A•B**x**C** has the following cyclic property:

$$\mathbf{A}\bullet\mathbf{B}\times\mathbf{C} = \mathbf{B}\bullet\mathbf{C}\times\mathbf{A} = \mathbf{C}\bullet\mathbf{A}\times\mathbf{B} \quad . \tag{2.20}$$

This identity is easily proved by referring to Figure 2-10, which shows a parallelepiped formed by the vectors **A**, **B**, and **C**.

Figure 2-10
A graphical depiction of the scalar triple product.



From solid geometry, the volume of the parallelepiped is $|\mathbf{A}||\mathbf{B}||\mathbf{C}|\sin\theta_1\cos\theta_2$, which can be expressed as **A•B×C**. Similar reasoning yields the other two expressions in equation 2.20.

In addition, the vector triple product **A×(B×C)** satisfies the following identity:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\bullet\mathbf{C}) - \mathbf{C}(\mathbf{A}\bullet\mathbf{B}) \quad . \qquad 2.21$$

This identity can be proven by expanding the vectors in the Cartesian coordinate (which will be discussed shortly).

2-4 Orthogonal Coordinate Systems

Our discussion of vectors has been hindered thus far by our inability to specify positions and directions, except through graphical representations. We will now introduce the concept of a coordinate system, which provides the framework necessary to describe these quantities without graphical representations.

Coordinate systems provide two attractive features that aid in vector operations. The first is the ability to specify positions in space by a sequence of scalars, called ***coordinates***. Coordinates identify the position of the point with respect to a coordinate center (or origin). The minimum number of scalars needed to uniquely specify a point in a particular domain (or space) determines the dimension of the space. Lines are one-dimensional, surfaces are two-dimensional, and volumes are three-dimensional.

Coordinate systems also provide a convenient way to specify vectors at any point. This is accomplished through the use of an orthogonal set of vectors, called ***base vectors***, which are defined at each point. Any vector can be uniquely defined in terms of its components in the base vector directions. The number of base vectors defined by a coordinate system equals the ***dimension*** of the space. Each base vector is defined at a point in space in terms of the position coordinates used to identify the point:

> In a coordinate system in which a point P is described by the coordinates $P(u_1,u_2,u_3)$, the i^{th} base unit vector $\hat{\mathbf{a}}_i$ at P has a direction parallel to a line through P along which only u_i varies, and points towards increasing values of u_i .

Three coordinate systems are discussed in this section: Cartesian (or rectangular), cylindrical, and spherical. Although there are many others, these three are sufficient to model all of the electromagnetic configurations discussed in this text. Why do we need more than one? The reason is that no one coordinate system is best suited for all situations.

2-4.1 The Cartesian Coordinate System

In the Cartesian coordinate system, three mutually perpendicular axes are used that intersect at a point, called the ***origin***. These axes are typically called the x, y, and z axes, respectively, and are oriented according to the right-hand rule: the rotation of the positive x axis into the positive y axis would cause a right-handed screw at the origin to thread along the positive z axis.

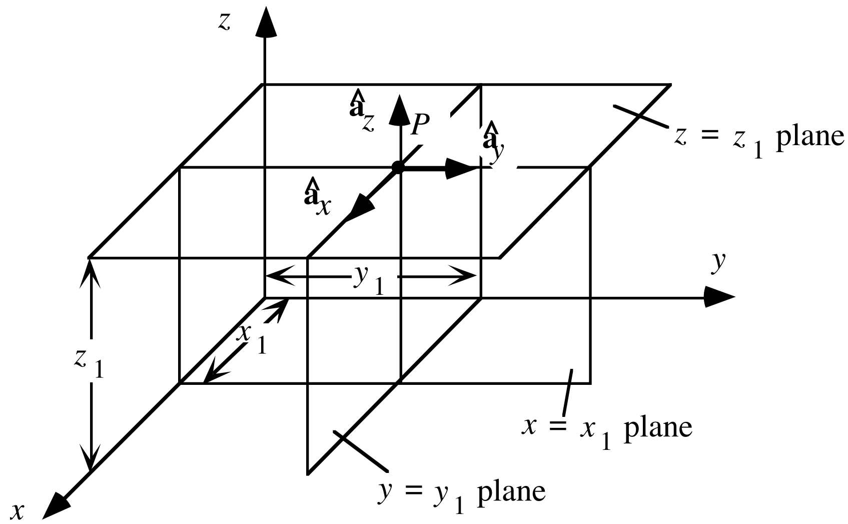


Figure 2-11
Position coordinates and base vectors in the Cartesian coordinate system.

In this coordinate system a point is identified by its three position coordinates: $u_1=x$, $u_2=y$, and $u_3=z$, each defined as the perpendicular distance from the point to the x, y, and z axes, respectively. As shown in Figure 2-11, any point can be envisioned as the point of intersection of three planes: x =constant, y=constant, and z=constant, where any of the three position coordinates can have any real value between $-\infty$ and $+\infty$.

The base vectors of the Cartesian coordinate system are particularly simple. At any point P, the unit vector $\hat{\mathbf{a}}_x$ is directed towards points having increasing values of x and perpendicular to the y=constant and z=constant planes. This direction is always parallel to the x axis, regardless of the location of P. The definitions of $\hat{\mathbf{a}}_y$ and $\hat{\mathbf{a}}_z$ are similar, and are shown in Figure 2-11. From these definitions it follows that the base vectors have the following product relationships:

$$\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_x = \hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_y = \hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_z = 1 \tag{2.22a}$$

$$\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_y = \hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_z = \hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_z = 0 \tag{2.22b}$$

$$\hat{\mathbf{a}}_x \times \hat{\mathbf{a}}_x = \hat{\mathbf{a}}_y \times \hat{\mathbf{a}}_y = \hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_z = 0 \tag{2.22c}$$

$$\hat{\mathbf{a}}_x \times \hat{\mathbf{a}}_y = \hat{\mathbf{a}}_z \tag{2.22d}$$

$$\hat{\mathbf{a}}_y \times \hat{\mathbf{a}}_z = \hat{\mathbf{a}}_x \tag{2.22e}$$

$$\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_x = \hat{\mathbf{a}}_y \quad . \tag{2.22f}$$

These product relations are simple to derive, but the cross products are somewhat difficult to remember. Fortunately, there is a simple way to remember them. Looking closely at these three relationships, we notice a sequence between the unit vectors that can be represented by the circle shown in Figure 2-12.

To determine the cross product between any two base vectors, start on the circle at the coordinate symbol of the first vector in the product and progress past the coordinate symbol of the second vector by the shortest route. The next symbol encountered along that route is the coordinate symbol of the resulting unit vector. The sign of this unit vector is positive if the progression is clockwise (i.e., along the arrows) and negative if it is counterclockwise.

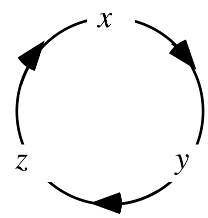


Figure 2-12
Circle diagram for cross products in Cartesian coordinates.

Any vector can be expanded at any point in terms of its components in the base vectors:

$$\mathbf{A} = A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z \quad , \qquad 2.23$$

where the scalars A_x, A_y, and A_z are the x, y, and z components of the vector **A**, respectively. Using equation 2.15, we can find these components by taking the dot product of both sides of equation 2.23 with each of the base vectors, yielding

$$A_i = \mathbf{A} \bullet \hat{\mathbf{a}}_i \qquad i = x, y, \text{or } z \quad . \qquad 2.24$$

Once the Cartesian components of two vectors are known, their scalar and vector products can be found without graphical representations. To accomplish this, we first express the dot product of **A** and **B** as

$$\mathbf{A} \bullet \mathbf{B} = (A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z) \bullet (B_x \hat{\mathbf{a}}_x + B_y \hat{\mathbf{a}}_y + B_z \hat{\mathbf{a}}_z) \ .$$

Using the orthogonality properties of the base vectors, this becomes

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad . \qquad 2.25$$

From equation 2.10, $|\mathbf{A}|$ can be expressed as

$$|\mathbf{A}| = \sqrt{\mathbf{A} \bullet \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2} \ , \qquad 2.26$$

which can also be derived from the Pythagorean Theorem.

Similarly, the cross product of **A** and **B** can be expressed in terms of components

$$\mathbf{A} \times \mathbf{B} = (A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z) \times (B_x \hat{\mathbf{a}}_x + B_y \hat{\mathbf{a}}_y + B_z \hat{\mathbf{a}}_z)$$

Using the cross product relations of the base vectors, this becomes

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\hat{\mathbf{a}}_x + (A_z B_x - A_x B_z)\hat{\mathbf{a}}_y + (A_x B_y - A_y B_x)\hat{\mathbf{a}}_z \ . \qquad 2.27$$

Each term in this formula can be evaluated using the circle aid in Figure 2-12. For instance, the term A_yB_z is positive because $\hat{\mathbf{a}}_y$ crossed into $\hat{\mathbf{a}}_z$ yields $+\hat{\mathbf{a}}_x$. Similarly, $\hat{\mathbf{a}}_z$ crossed into $\hat{\mathbf{a}}_y$ yields $-\hat{\mathbf{a}}_x$, so the sign of the term A_zB_y is negative. Equation 2.27 can also be written as a determinant,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} . \qquad 2.28$$

Expanding this determinant by minors yields

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\mathbf{a}}_x - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{\mathbf{a}}_y + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{\mathbf{a}}_z . \qquad 2.29$$

Another attractive feature of coordinate systems is that they provide simple expressions for the differential quantities needed to evaluate integrals of vector and scalar fields. Figure 2-13 shows the differential volume traced about a point when its position coordinates of *x, y*, and *z* are varied by the differential amounts *dx*, *dy*, and *dz*, respectively.

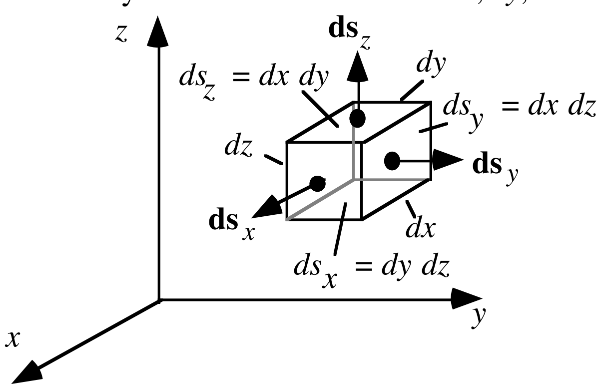


Figure 2-13
Differential volume and surface elements in the Cartesian coordinate system.

The enclosed volume *dv* is given by

$$dv = dx\, dy\, dz \; . \qquad 2.30$$

Figure 2-13 also shows three differential surfaces traced when two coordinates at a point are varied by differential amounts and the third is held constant. Each of the surfaces is named according to the direction of its ***normal*** (i.e., perpendicular) direction, which is defined as the unit vector that is perpendicular to each vector (or line) that lies on that surface. From Figure 2-13 we see that the normal to each of these surfaces is the base vector corresponding to the coordinate that is constant on the surface. The area of each differential surface is

$$ds_x = dy\, dz \quad (\text{when } dx = 0) \qquad 2.31a$$

$$ds_y = dx\, dz \quad (\text{when } dy = 0) \qquad 2.31b$$

$$ds_z = dx\, dy \quad (\text{when } dz = 0) . \qquad 2.31c$$

We can also define ***differential surface vectors*** for each of these differential surfaces. The magnitude of each differential surface vector equals the differential surface area, and its direction is normal to the surface. The three differential surface vectors shown in Figure 2-13 can be expressed as

$$\mathbf{ds}_x = ds_x\, \hat{\mathbf{a}}_x = dy\ dz\, \hat{\mathbf{a}}_x \quad \text{(when } dx = 0\text{)} \qquad 2.32a$$

$$\mathbf{ds}_y = ds_y\, \hat{\mathbf{a}}_y = dx\ dz\, \hat{\mathbf{a}}_y \quad \text{(when } dy = 0\text{)} \qquad 2.32b$$

$$\mathbf{ds}_z = ds_z\, \hat{\mathbf{a}}_z = dx\ dy\, \hat{\mathbf{a}}_z \quad \text{(when } dz = 0\text{).} \qquad 2.32c$$

Notice in these expressions that of the two possible normal directions for each surface, the direction outward from the enclosed volume is chosen in each case. This convention is always followed whenever a differential surface is part of a larger surface that completely encloses a volume. Such surfaces are called ***closed surfaces***.

When integrating along a line of points, it is necessary to define a differential vector that represents the magnitude and direction of each segment of the path. Consider the path shown in Figure 2-14.

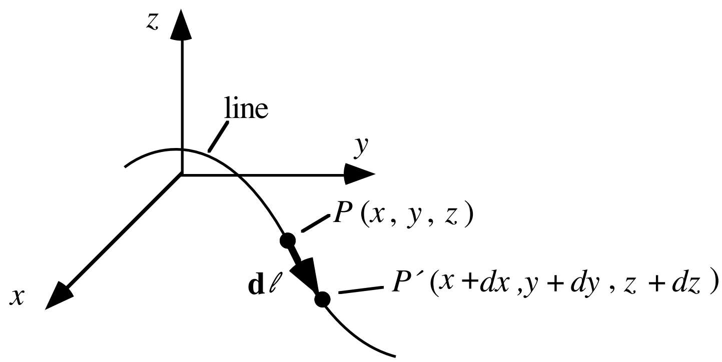


Figure 2-14
A differential displacement vector **d**ℓ along an arbitrary path (line).

We define the ***differential displacement vector*** **d**ℓ at point $P(x,y,z)$ to be the directed distance from $P(x,y,z)$ to $P'(x+dx, y+dy, z+dz)$. Along any path, the differential displacement vector can be represented as

$$\mathbf{d}\ell = dx\, \hat{\mathbf{a}}_x + dy\, \hat{\mathbf{a}}_y + dz\, \hat{\mathbf{a}}_z \quad . \qquad 2.33$$

Here, it is important to note that dx, dy, and dz are *not* independent quantities, since each is a measure of how rapidly the x, y, and z coordinates are varying at the point, respectively.

A frequently used method of finding dx, dy, and dz is to write the position coordinates of the line using a single, common variable, called a ***parametric variable***. Thus, when a line can be represented by $P[x(s),y(s),z(s)]$), where $x(s)$, $y(s)$, and $z(s)$ are functions of the parametric variable s, the differentials dx, dy, and dz can be obtained from the relations

$$dx = \frac{dx(s)}{ds}\, ds \qquad 2.34a$$

$$dy = \frac{dy(s)}{ds}\, ds \qquad 2.34b$$

$$dz = \frac{dz(s)}{ds}\, ds \ . \qquad 2.34c$$

Example 2-1

For the vectors $\mathbf{A} = -\hat{\mathbf{a}}_x - 2\hat{\mathbf{a}}_y + 4\hat{\mathbf{a}}_z$ and $\mathbf{B} = -\hat{\mathbf{a}}_x + 3\hat{\mathbf{a}}_y - 2\hat{\mathbf{a}}_z$, find the smallest angle θ_{AB} between **A** and **B** and the unit vector $\hat{\mathbf{a}}_n$ that points along the direction of **A**x**B**.

Solution:

We can find θ_{AB} by using the dot product. Solving equation 2.8 for θ_{AB}, we have:

$$\theta_{AB} = \cos^{-1} \frac{\mathbf{A} \bullet \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \quad .$$

Using equations 2.25 and 2.26, we have

$$\mathbf{A} \bullet \mathbf{B} = (-1)(-1) + (-2)(3) + (4)(-2) = -13$$

$$|\mathbf{A}| = \sqrt{(-1)^2 + (-2)^2 + (4)^2} = \sqrt{21}$$

$$|\mathbf{B}| = \sqrt{(-1)^2 + (3)^2 + (-2)^2} = \sqrt{14} \quad .$$

Substituting, we find

$$\theta_{AB} = \cos^{-1} \frac{-13}{\sqrt{21}\,\sqrt{14}} = 139.3°.$$

To find $\hat{\mathbf{a}}_n$, we first solve equation 2.16 for $\hat{\mathbf{a}}_n$, yielding

$$\hat{\mathbf{a}}_n = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A}||\mathbf{B}|\sin\theta_{AB}} \quad .$$

We can find **A**x**B** by using equation 2.27:

$$\mathbf{A} \times \mathbf{B} = (4-12)\hat{\mathbf{a}}_x + (-4-2)\hat{\mathbf{a}}_y + (-3-2)\hat{\mathbf{a}}_z = -8\hat{\mathbf{a}}_x - 6\hat{\mathbf{a}}_y - 5\hat{\mathbf{a}}_z \quad .$$

Substituting this into the expression for $\hat{\mathbf{a}}_n$, we obtain

$$\hat{\mathbf{a}}_n = \frac{-8\hat{\mathbf{a}}_x - 6\hat{\mathbf{a}}_y - 5\hat{\mathbf{a}}_z}{\sqrt{21}\sqrt{14}\,\sin(139.3°)} = -0.716\,\hat{\mathbf{a}}_x - 0.537\,\hat{\mathbf{a}}_y - 0.447\,\hat{\mathbf{a}}_z \quad .$$

Example 2-2

Find an expression for the differential displacement vector $\mathbf{d}\ell$ at any point on the half-circle path shown in Figure 2-15. Assume that the direction along the circle is counterclockwise.

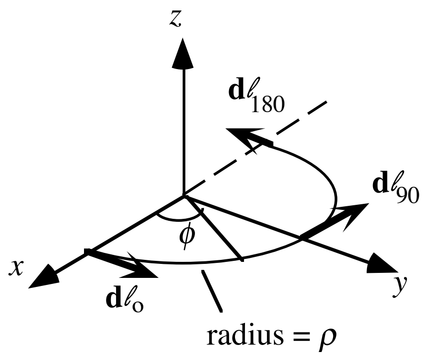


Figure 2-15
Differential displacement vectors along a counterclockwise, semicircular path of radius *a*.

Solution:

Since the half circle has a unit radius, the position coordinates (x,y) can be written in terms of the parametric variable ϕ,

$$x = \rho \cos\phi$$

$$y = \rho \sin\phi \ .$$

Using equations 2.34a and 2.34b, we find

$$dx = \frac{dx}{d\phi} d\phi \ = -\rho \sin\phi \, d\phi$$

and

$$dy = \frac{dy}{d\phi} d\phi \ = \rho \cos\phi \, d\phi \ .$$

Hence, the differential displacement vector at any point can be written as

$$\mathbf{d}\ell = \rho \, (-\sin\phi \, \hat{\mathbf{a}}_x + \cos\phi \, \hat{\mathbf{a}}_y) d\phi \ .$$

To see if this result makes any sense, let us evaluate $\mathbf{d}\ell$ at the points $\phi = 0$, 90°, and 180°. Substituting, we obtain

$$\mathbf{d}\ell_o = \ \hat{\mathbf{a}}_y \rho \, d\phi \quad \text{at } \phi = 0 \ ,$$

$$\mathbf{d}\ell_{90} = - \hat{\mathbf{a}}_x \rho \, d\phi \quad \text{at } \phi = 90° \ ,$$

$$\mathbf{d}\ell_{180} = - \hat{\mathbf{a}}_y \rho \, d\phi \quad \text{at } \phi = 180° \ .$$

These vectors are shown with amplified lengths (so that they can be seen) in Figure 2-15. Notice that each vector is tangent to the circle and has magnitude $\rho \, d\phi$.

2-4.2 The Cylindrical Coordinate System

The cylindrical coordinate system is a three-dimensional version of the polar coordinate system used in two-dimensional analysis[4]. Referring to Figure 2-16, the position coordinates of a point P in this system are $u_1=\rho$, $u_2=\phi$, and $u_3=z$. Here, ρ is defined as the perpendicular projection from the point to the z axis and ϕ is the angle that this projection makes with respect to the x axis. The z coordinate is the same as in Cartesian coordinates. All points are uniquely specified by the intersection of ρ=constant, ϕ=constant, and z=constant surfaces, where $0 < \rho < \infty$, $0 < \phi < 2\pi$, and $-\infty < z < \infty$.

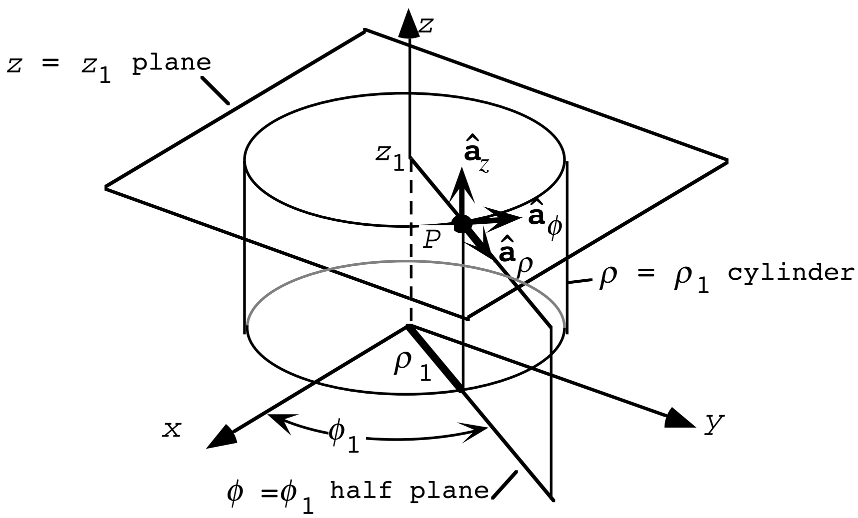


Figure 2-16
Position coordinates and base vectors in the cylindrical coordinate system.

Using Figure 2-16, it is simple for one to show that cylindrical and Cartesian coordinates are related by

$$\rho = \sqrt{x^2 + y^2} \qquad 2.35a$$

$$\phi = \tan^{-1}\left\{\frac{y}{x}\right\} \qquad 2.35b$$

$$z = z \qquad 2.35c$$

and

$$x = \rho \cos\phi \qquad 2.36a$$

$$y = \rho \sin\phi \qquad 2.36b$$

$$z = z \quad . \qquad 2.36c$$

[4]There are many different cylindrical coordinate systems, such as circular cylindrical coordinates, elliptical cylindrical coordinates, and parabolic cylindrical coordinates. Throughout this text, however, we will refer to the circular cylindrical coordinate system as simply the cylindrical coordinate system.

Care must be taken when using equation 2.35b since $0<\phi<2\pi$, and the $\tan^{-1}$ function has a principal-value range of $-\frac{\pi}{2}<\phi<\frac{\pi}{2}$. Because of this, π must be added[5] to the value of ϕ specified by equation 2.35b when a point lies in the second or third quadrants (i.e., $x<0$).

The base unit vectors of the cylindrical coordinate system, $\hat{\mathbf{a}}_\rho$, $\hat{\mathbf{a}}_\phi$, and $\hat{\mathbf{a}}_z$, are depicted in Figure 2-16. These vectors are directed towards increasing values of ρ, ϕ, and z, respectively, and are perpendicular to the constant-coordinate surfaces of the other coordinates. Unlike the Cartesian coordinate system, where all three base vectors maintain the same orientations at all points, two of the base vectors in the cylindrical coordinate system vary with the coordinate ϕ; one must first define the ϕ coordinate of a point before the $\hat{\mathbf{a}}_\rho$ and $\hat{\mathbf{a}}_\phi$ directions can be specified.

From basic trigonometry, the following relationships can be derived,

$$\hat{\mathbf{a}}_\rho \bullet \hat{\mathbf{a}}_\rho = \hat{\mathbf{a}}_\phi \bullet \hat{\mathbf{a}}_\phi = \hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_z = 1 \qquad 2.37a$$

$$\hat{\mathbf{a}}_\rho \bullet \hat{\mathbf{a}}_\phi = \hat{\mathbf{a}}_\rho \bullet \hat{\mathbf{a}}_z = \hat{\mathbf{a}}_\phi \bullet \hat{\mathbf{a}}_z = 0 \qquad 2.37b$$

$$\hat{\mathbf{a}}_\rho \times \hat{\mathbf{a}}_\rho = \hat{\mathbf{a}}_\phi \times \hat{\mathbf{a}}_\phi = \hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_z = 0 \qquad 2.37c$$

$$\hat{\mathbf{a}}_\rho \times \hat{\mathbf{a}}_\phi = \hat{\mathbf{a}}_z \qquad 2.37d$$

$$\hat{\mathbf{a}}_\phi \times \hat{\mathbf{a}}_z = \hat{\mathbf{a}}_\rho \qquad 2.37e$$

$$\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_\rho = \hat{\mathbf{a}}_\phi \quad . \qquad 2.37f$$

The cross products between the cylindrical base vectors can be symbolized using the aid shown in Figure 2-17. A vector **A** at any point can be represented by its components in the base vectors at that point:

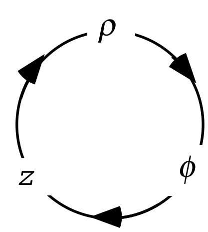


Figure 2-17
Circle diagram for cross products in cylindrical coordinates

$$\mathbf{A}=A_\rho \hat{\mathbf{a}}_\rho + A_\phi \hat{\mathbf{a}}_\phi + A_z \hat{\mathbf{a}}_z \quad , \qquad 2.38$$

where the scalars A_ρ, A_ϕ, A_z are the ρ, ϕ, and z components of **A**, respectively. Using equation 2.15, these components can be found by taking the dot product of **A** with each of the base vectors:

---

[5]Most calculators have a polar-to-rectangular function that automatically performs this function when x and y are specified separately.

$$A_i = \mathbf{A} \bullet \hat{\mathbf{a}}_i \qquad i = \rho, \phi, \text{ or } z \quad . \qquad 2.39$$

The dot product of two vectors, **A** and **B**, can be expressed in terms of their components,

$$\mathbf{A}\bullet\mathbf{B} = (A_\rho \hat{\mathbf{a}}_\rho + A_\phi \hat{\mathbf{a}}_\phi + A_z \hat{\mathbf{a}}_z)\bullet(B_\rho \hat{\mathbf{a}}_\rho + B_\phi \hat{\mathbf{a}}_\phi + B_z \hat{\mathbf{a}}_z) \; .$$

Using the orthogonality properties of the base vectors, this becomes

$$\mathbf{A}\bullet\mathbf{B} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z \quad . \qquad 2.40$$

Since $|\mathbf{A}| = \sqrt{\mathbf{A}\bullet\mathbf{A}}$, it follows that

$$|\mathbf{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2} \quad , \qquad 2.41$$

which can also be derived from the Pythagorean Theorem.

Similarly, the cross product of two vectors can be expressed as

$$\mathbf{A} \times \mathbf{B} = (A_\rho \hat{\mathbf{a}}_\rho + A_\phi \hat{\mathbf{a}}_\phi + A_z \hat{\mathbf{a}}_z) \times (B_\rho \hat{\mathbf{a}}_\rho + B_\phi \hat{\mathbf{a}}_\phi + B_z \hat{\mathbf{a}}_z) \; ,$$

which, using the properties of the base vectors, can be simplified to read

$$\mathbf{A}\times\mathbf{B} = (A_\phi B_z - A_z B_\phi)\hat{\mathbf{a}}_\rho + (A_z B_\rho - A_\rho B_z)\hat{\mathbf{a}}_\phi + (A_\rho B_\phi - A_\phi B_\rho)\hat{\mathbf{a}}_z \; . \qquad 2.42$$

This can also be written in shorthand as the determinant

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{a}}_\rho & \hat{\mathbf{a}}_\phi & \hat{\mathbf{a}}_z \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix} \quad . \qquad 2.43$$

As shown in Figure 2-18, a differential volume *dv* is traced about a point when its coordinates are varied by the amounts $d\rho$, $d\phi$, and dz, respectively.

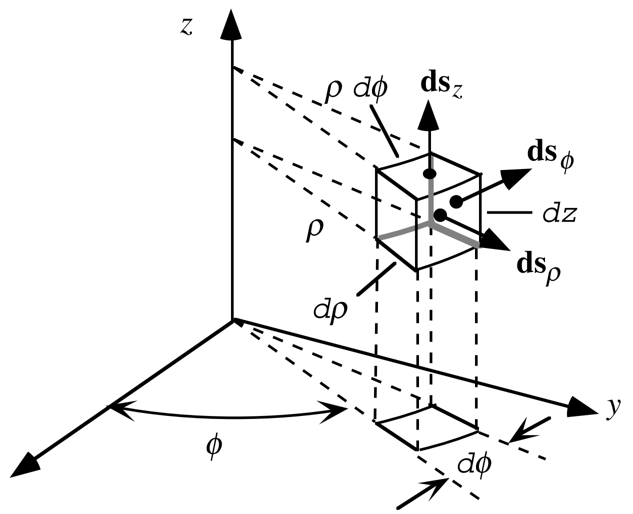


Figure 2-18
Differential volume and surface elements in the cylindrical coordinate system.

In the limit as $d\rho$, $d\phi$, and dz approach zero, the enclosed volume dv can be written as

$$dv = \rho d\rho \, d\phi dz \quad . \qquad 2.44$$

Notice that the factor ρ is necessary because the distance traced as the angular coordinate varies from ϕ to $\phi + d\phi$ equals $\rho d\phi$. This factor also makes the equation dimensionally correct since, strictly speaking, $d\phi$ is unitless[6].

As can be deduced from Figure 2-18, the surface areas traced when two of the three coordinates at a point vary by differential amounts are

$$ds_\rho = \rho d\phi \, dz \quad (\text{when } d\rho = 0) \qquad 2.45a$$

$$ds_\phi = d\rho \, dz \quad (\text{when } d\phi = 0) \qquad 2.45b$$

$$ds_z = \rho \, d\rho \, d\phi \quad (\text{when } dz = 0) \quad . \qquad 2.45c$$

The differential surface vectors associated with these surfaces are found from these by adding the appropriate unit vectors,

$$\mathbf{ds}_\rho = ds_\rho \hat{\mathbf{a}}_\rho = \rho d\phi \, dz \, \hat{\mathbf{a}}_\rho \quad (\text{when } d\rho = 0) \qquad 2.46a$$

$$\mathbf{ds}_\phi = ds_\phi \hat{\mathbf{a}}_\phi = d\rho \, dz \, \hat{\mathbf{a}}_\phi \quad (\text{when } d\phi = 0) \qquad 2.46b$$

$$\mathbf{ds}_z = ds_z \hat{\mathbf{a}}_z = \rho \, d\rho \, d\phi \, \hat{\mathbf{a}}_z \quad (\text{when } dz = 0) \quad . \qquad 2.46c$$

Finally, the differential displacement vector that represents the directed distance from $P(\rho,\phi,z)$

---

[6]The units of ϕ and $d\phi$ are radians (or degrees), but the radian is defined as the ratio of arc length to the circumference of a circle, so it is actually unitless. The same is true for the degree.

to $P'(\rho+d\rho, \phi+d\phi, z+dz)$ along a line contour is

$$\mathbf{d}\ell = d\rho\, \hat{\mathbf{a}}_\rho + \rho\, d\phi\, \hat{\mathbf{a}}_\phi + dz\, \hat{\mathbf{a}}_z \quad . \qquad 2.47$$

If the coordinates along the line are defined by $\rho(s)$, $\phi(s)$, and $z(s)$, then $d\rho$, $d\phi$, and dz can be found from the relations

$$d\rho = \frac{d\rho(s)}{ds}\, ds \qquad 2.48a$$

$$d\phi = \frac{d\phi(s)}{ds}\, ds \qquad 2.48b$$

$$dz = \frac{dz(s)}{ds}\, ds \quad . \qquad 2.48c$$

2-4.3 The Spherical Coordinate System

The spherical coordinate system identifies points by the coordinates $u_1=r$, $u_2=\theta$, and $u_3=\phi$, where r is the length of the line extending from the origin to the point, θ is the angle that this line makes with the z axis, and ϕ has the same definition as in cylindrical coordinates. As shown in Figure 2-19, any point is uniquely defined as the point of intersection of the r=constant, θ=constant, and ϕ=constant surfaces, where $0 < r < \infty$, $0 < \theta < \pi$, and $0 < \phi < 2\pi$.

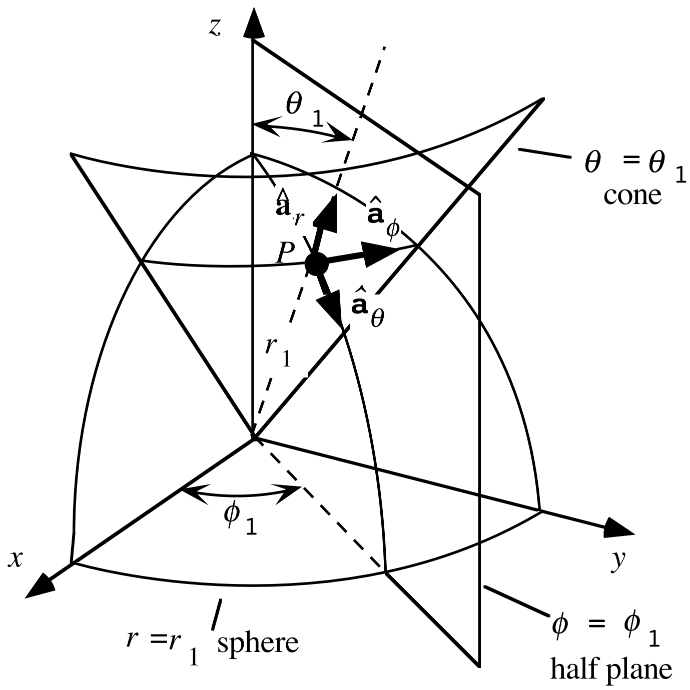


Figure 2-19
Position coordinates and base vectors in the spherical coordinate system.

The spherical and Cartesian coordinates of a point are related by:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad 2.49a$$

$$\theta = \cos^{-1}\left\{\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right\} \qquad 2.49b$$

$$\phi = \tan^{-1}\left\{\frac{y}{x}\right\} \qquad 2.49c$$

and

$$x = r\sin\theta\cos\phi \qquad 2.50a$$

$$y = r\sin\theta\sin\phi \qquad 2.50b$$

$$z = r\cos\theta\ . \qquad 2.50c$$

As in the case of cylindrical coordinates, care must be exercised when using equation 2.49c to ensure that the calculated angle ϕ lies in the correct quadrant.

The base unit vectors in the cylindrical coordinate system, $\hat{\mathbf{a}}_r$, $\hat{\mathbf{a}}_\theta$, and $\hat{\mathbf{a}}_\phi$, are directed towards increasing values of r, θ, and ϕ, respectively. As can be seen from Figure 2-19, all three of these vectors are functions of the coordinates θ and ϕ. Thus, the coordinates of a point must be specified before the base unit vectors can be specified.

At any point, a vector **A** can be expressed in terms of its components in the base vector directions:

$$\mathbf{A} = A_r\,\hat{\mathbf{a}}_r + A_\theta\,\hat{\mathbf{a}}_\theta + A_\phi\,\hat{\mathbf{a}}_\phi \quad , \qquad 2.51$$

where A_r, A_θ, and A_ϕ are the r, θ, and ϕ components of **A**, respectively. These components can be found via the dot product

$$A_i = \mathbf{A} \bullet \hat{\mathbf{a}}_i \qquad i = r, \theta, \text{ or } \phi \quad . \qquad 2.52$$

The base vectors of the spherical coordinate system satisfy the following relationships:

$$\hat{\mathbf{a}}_r \bullet \hat{\mathbf{a}}_r = \hat{\mathbf{a}}_\theta \bullet \hat{\mathbf{a}}_\theta = \hat{\mathbf{a}}_\phi \bullet \hat{\mathbf{a}}_\phi = 1 \qquad 2.53a$$

$$\hat{\mathbf{a}}_r \bullet \hat{\mathbf{a}}_\theta = \hat{\mathbf{a}}_r \bullet \hat{\mathbf{a}}_\phi = \hat{\mathbf{a}}_\theta \bullet \hat{\mathbf{a}}_\phi = 0 \qquad 2.53b$$

$$\hat{\mathbf{a}}_r \times \hat{\mathbf{a}}_r = \hat{\mathbf{a}}_\theta \times \hat{\mathbf{a}}_\theta = \hat{\mathbf{a}}_\phi \times \hat{\mathbf{a}}_\phi = 0 \qquad 2.53c$$

$$\hat{\mathbf{a}}_r \times \hat{\mathbf{a}}_\theta = \hat{\mathbf{a}}_\phi \qquad 2.53d$$

$$\hat{\mathbf{a}}_\theta \times \hat{\mathbf{a}}_\phi = \hat{\mathbf{a}}_r \qquad 2.53e$$

$$\hat{\mathbf{a}}_\phi \times \hat{\mathbf{a}}_r = \hat{\mathbf{a}}_\theta \ . \qquad 2.53f$$

The cross product relationships between the spherical base vectors can be symbolized using the aid shown in Figure 2-20. The dot product of any two vectors can be expressed as:

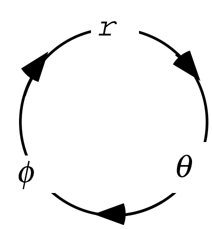


Figure 2-20

Circle diagram for cross products in spherical coordinates

$$\mathbf{A}\bullet\mathbf{B} = (A_r\,\hat{\mathbf{a}}_r + A_\theta\,\hat{\mathbf{a}}_\theta + A_\phi\,\hat{\mathbf{a}}_\phi)\bullet(B_r\,\hat{\mathbf{a}}_r + B_\theta\,\hat{\mathbf{a}}_\theta + B_\phi\,\hat{\mathbf{a}}_\phi)$$

$$= A_r B_r + A_\theta B_\theta + A_\phi B_\phi \quad . \qquad 2.54$$

Also, since $|\mathbf{A}| = \sqrt{\mathbf{A}\bullet\mathbf{A}}$, it follows that

$$|\mathbf{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2} \quad , \qquad 2.55$$

which can also be derived from the Pythagorean Theorem.

Similarly, the cross product of two vectors can be expressed as

$$\mathbf{A} \times \mathbf{B} = (A_r \hat{\mathbf{a}}_r + A_\theta \hat{\mathbf{a}}_\theta + A_\phi \hat{\mathbf{a}}_\phi) \times (B_r \hat{\mathbf{a}}_r + B_\theta \hat{\mathbf{a}}_\theta + B_\phi \hat{\mathbf{a}}_\phi),$$

which reduces to

$$\mathbf{A} \times \mathbf{B} = (A_\theta B_\phi - A_\phi B_\theta)\hat{\mathbf{a}}_r + (A_\phi B_r - A_r B_\phi)\hat{\mathbf{a}}_\theta + (A_r B_\theta - A_\theta B_r)\hat{\mathbf{a}}_\phi \qquad 2.56$$

This expression can be written in shorthand form as the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{a}}_r & \hat{\mathbf{a}}_\theta & \hat{\mathbf{a}}_\phi \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix} . \qquad 2.57$$

As shown in Figure 2-21, a differential volume dv is traced about a point $P(r,\theta,\phi)$ when its coordinates are varied by the amounts dr, $d\theta$, and $d\phi$, respectively.

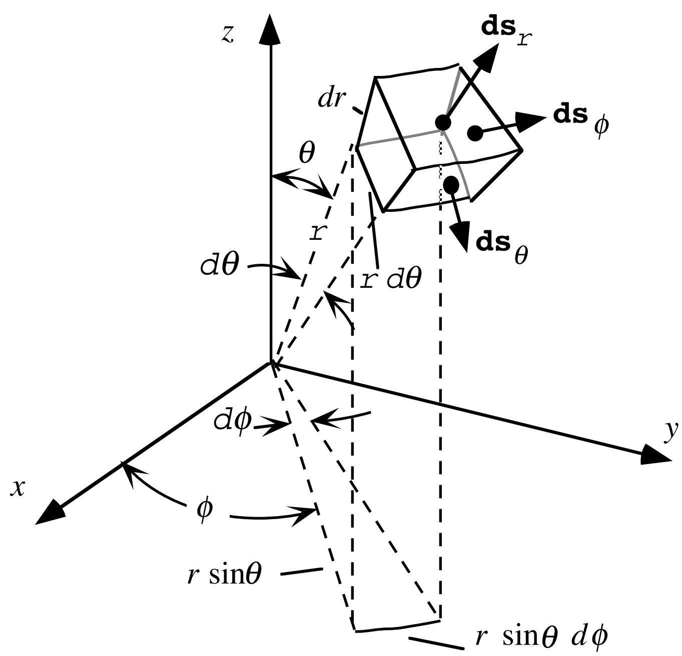


Figure 2-21
Differential volume and surface elements in the spherical coordinate system.

In the limit as dr, $d\phi$, and dz approach zero, the enclosed volume is

$$dv = r^2 \sin\theta \, dr \, d\theta d\phi \ . \qquad 2.58$$

In this expression, the multiplier $r^2 \sin\theta$ is necessary because the lengths traced by differential changes in θ and ϕ are $rd\theta$ and $r \sin\theta d\phi$, respectively. Also, the r^2 makes the expression dimensionally correct.

The surfaces traced when two of the three coordinates are varied by differential amounts are also shown in Figure 2-21. They have areas given by

$$ds_r = r^2 \sin\theta \, d\theta \, d\phi \quad \text{(when } dr = 0\text{)} \qquad 2.59a$$

$$ds_\theta = r \sin\theta \, dr \, d\phi \quad \text{(when } d\theta = 0\text{)} \qquad 2.59b$$

$$ds_\phi = r \, dr \, d\theta \quad \text{(when } d\phi = 0\text{)} \ , \qquad 2.59c$$

and their associated differential surface vectors are

$$\mathbf{ds}_r = ds_r \hat{\mathbf{a}}_r = r^2 \sin\theta \, d\theta \, d\phi \ \hat{\mathbf{a}}_r \quad \text{(when } dr = 0\text{)} \qquad 2.60a$$

$$\mathbf{ds}_\theta = ds_\theta \hat{\mathbf{a}}_\theta = r \sin\theta \, dr \, d\phi \ \hat{\mathbf{a}}_\theta \quad \text{(when } d\theta = 0\text{)} \qquad 2.60b$$

$$\mathbf{ds}_\phi = ds_\phi \hat{\mathbf{a}}_\phi = r \, dr \, d\theta \, \hat{\mathbf{a}}_\phi \quad \text{(when } d\phi = 0\text{)} \ . \qquad 2.60c$$

Finally, the differential displacement vector that represents the directed distance from $P(r,\theta,\phi)$ to $P'(r+dr,\theta+d\theta,\phi+d\phi)$ along a line contour is

$$\mathbf{d\ell} = dr \, \hat{\mathbf{a}}_r + r \, d\theta \, \hat{\mathbf{a}}_\theta + r \sin\theta d\phi \, \hat{\mathbf{a}}_\phi \ . \qquad 2.61$$

If the coordinates along the line are given by $r(s)$, $\theta(s)$, and $\phi(s)$, then dr, $d\theta$, and $d\phi$ can be found from

$$dr = \frac{dr(s)}{ds} ds \qquad 2.62a$$

$$d\theta = \frac{d\theta(s)}{ds} ds \qquad 2.62b$$

$$d\phi = \frac{d\phi(s)}{ds} ds \ . \qquad 2.62c$$

2-4.4 Conversions between coordinate systems

There are many times when it is necessary to change the representation of a vector from one coordinate system to another. Typically, this is done when different aspects of a problem are most easily described using different coordinate representations.

Changing a vector's representation from one coordinate system to another requires two steps:

- Convert the coordinates
- Convert the components

The position coordinates in the new system are found simply by applying the appropriate coordinate transformations. The components in the new system are found by taking the dot product of the vector with each of the base vectors in the new system,

$$A_i = \mathbf{A} \bullet \hat{\mathbf{a}}_i \quad , \qquad 2.63$$

where i is set equal to each of the coordinate variables in the new system.

Appendix B contains a number of tables that are helpful when converting the representation of a vector from one coordinate system to another. Table B-1 contains the relationships between the coordinate variables of the three coordinate systems. Table B-2 contains the dot products of the base vectors of the three coordinate systems. Finally, Table B-3 summarizes the relationships between vector components in these coordinate systems.

Example 2-3

Find the representation of $\mathbf{C} = \rho \hat{\mathbf{a}}_\phi$ in Cartesian coordinates.

Solution:

Using equation 2.63 in conjunction with the values in Table B-2, the Cartesian components of **C** are

$$C_x = \mathbf{C} \bullet \hat{\mathbf{a}}_x = \rho\, \hat{\mathbf{a}}_\phi \bullet \hat{\mathbf{a}}_x = -\rho \sin\phi$$

$$C_y = \mathbf{C} \bullet \hat{\mathbf{a}}_y = \rho\, \hat{\mathbf{a}}_\phi \bullet \hat{\mathbf{a}}_y = \rho \cos\phi$$

$$C_z = \mathbf{C} \bullet \hat{\mathbf{a}}_z = \rho\, \hat{\mathbf{a}}_\phi \bullet \hat{\mathbf{a}}_z = 0 \quad .$$

Next, using $x = \rho \cos\phi$ and $y = \rho \sin\phi$, we obtain

$$\mathbf{C} = C_x \hat{\mathbf{a}}_x + C_y \hat{\mathbf{a}}_y + C_z \hat{\mathbf{a}}_z = -\rho \sin\phi\ \hat{\mathbf{a}}_x + \rho \cos\phi\, \hat{\mathbf{a}}_y$$

$$= -y\, \hat{\mathbf{a}}_x + x\, \hat{\mathbf{a}}_y \quad .$$

Example 2-4

Find the representation of $\mathbf{A} = 3\, y\ \hat{\mathbf{a}}_x$ in the spherical coordinate system.

Solution:

Knowing that $y = r \sin\theta \sin\phi$, we can write **A** as

$$\mathbf{A} = 3\, r \sin\theta \sin\phi\ \hat{\mathbf{a}}_x \quad .$$

Thus, $A_x = 3\, r \sin\theta \sin\phi$, $A_y = A_z = 0$.

Using table B-3, the spherical components of **A** are

$$A_r = A_x \sin\theta \cos\phi = 3r \sin^2\theta \sin\phi \cos\phi$$

$$A_\theta = A_x \cos\theta \cos\phi = 3r \sin\theta \cos\theta \sin\phi \cos\phi$$
$$= \frac{3}{4} r \sin 2\theta \sin 2\phi$$

$$A_\phi = - A_x \sin\phi = -3 r \sin\theta \sin^2 \phi \ .$$

Thus, the representation of **A** in spherical coordinates is

$$\mathbf{A} = 3r\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{a}}_r + \frac{3}{4}\,r\sin 2\theta\sin 2\phi\,\hat{\mathbf{a}}_\theta - 3r\sin\theta\sin^2\phi\,\hat{\mathbf{a}}_\phi\ .$$

2-4.5 The Position Vector

We have already seen that any point is uniquely defined by its position coordinates. We can also identify a point by its ***position vector***.

> The position vector of a point is defined as the directed distance from the origin to the point and is represented by the symbol **r**.

Every point has a unique position vector that identifies it. This vector is denoted by the symbol **r** and is depicted in Figure 2-22 for an arbitrary point *P*. The position vector of an arbitrary point has the following representations in the Cartesian, cylindrical, and spherical coordinate systems:

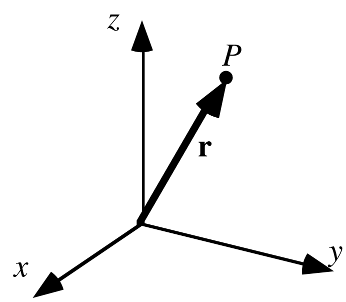


Figure 2-22

The position vector

$$\mathbf{r} = \begin{cases} x\,\hat{\mathbf{a}}_x + y\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z & \text{Cartesian} \quad 2.64\text{a} \\ \rho\,\hat{\mathbf{a}}_\rho + z\,\hat{\mathbf{a}}_z & \text{cylindrical} \quad 2.64\text{b} \\ r\,\hat{\mathbf{a}}_r & \text{spherical} \quad 2.64\text{c} \end{cases}$$

The spherical representation follows directly from the definition of **r**. The other two representations can be found through coordinate transformations.

The directed distance between two points can be expressed in terms of their position vectors. Referring to Figure 2-23, consider the points *P* and *P*´, represented in Figure 2-22 by the position vectors **r** and **r**´, respectively. If we let **R** denote the directed distance from *P*´ to *P*, it follows from Figure 2-23 that

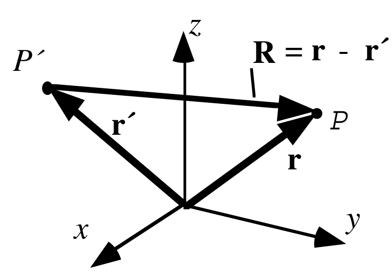


Figure 2-23

The directed distance **R** between two arbitrary points

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'.$$

We can also write **R** as:

$$\mathbf{R} = R\,\hat{\mathbf{a}}_R\ , \tag{2.65}$$

where,

$$R = |\mathbf{r}\text{-}\mathbf{r}'| \tag{2.66}$$

and

$$\hat{\mathbf{a}}_R = \frac{\mathbf{r}\text{-}\mathbf{r}'}{|\mathbf{r}\text{-}\mathbf{r}'|}\ . \tag{2.67}$$

In Cartesian coordinates, **R** is represented by:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = (x\text{-}x')\hat{\mathbf{a}}_x + (y\text{-}y')\hat{\mathbf{a}}_y + (z\text{-}z')\hat{\mathbf{a}}_z\ . \tag{2.68}$$

In coordinate systems other than Cartesian coordinates, it is important to remember that the base unit vectors at **r** and **r´** are not, in general, the same. This is demonstrated in the following example.

Example 2-5

Find the directed distance from $P'(3,30°,1)$ to $P(1,90°,2)$. Write the representation of this vector at both points using the Cylindrical coordinate system.

Solution:

Since the values of ϕ at these two points are different, it is best to start by finding $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ in the Cartesian coordinate system. Converting P and P' to Cartesian coordinates

$$P(1,90°,2) = P(\cos 90°, \sin 90°, 2) = P(0,1,2)$$

$$P'(3,30°,1) = P'(3\cos 30°, 3\sin 30°,1) = P'(\ 2.66,1.5,1)\ .$$

The corresponding position vectors are

$$\mathbf{r} = \hat{\mathbf{a}}_y + 2\hat{\mathbf{a}}_z$$

$$\mathbf{r}' = 2.66\hat{\mathbf{a}}_x + 1.5\,\hat{\mathbf{a}}_y + \hat{\mathbf{a}}_z\ ,$$

so the directed distance from P' to P is

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = -2.66\hat{\mathbf{a}}_x - 0.5\hat{\mathbf{a}}_y + \hat{\mathbf{a}}_z\ .$$

Using Table B-3, we can express **R** in cylindrical coordinates at any point as

$$\mathbf{R} = R_\rho\,\hat{\mathbf{a}}_\rho + R_\phi\,\hat{\mathbf{a}}_\phi + R_z\,\hat{\mathbf{a}}_z\ ,$$

where

$$R_\rho = -2.66\cos\phi - 0.5\sin\phi$$

$$R_\phi = 2.66\sin\phi - 0.5\cos\phi$$

$$R_z = 1\ ,$$

and ϕ is the position coordinate at the point. Substituting, $\phi = 90°$ at P and $\phi = 30°$ at P', we find

$$\mathbf{R} = -0.5\,\hat{\mathbf{a}}_\rho + 2.66\,\hat{\mathbf{a}}_\phi - \hat{\mathbf{a}}_z \qquad \text{at } P$$

and

$$\mathbf{R} = -2.56\,\hat{\mathbf{a}}_{\rho'} + 0.9\,\hat{\mathbf{a}}_{\phi'} - \hat{\mathbf{a}}_z \qquad \text{at } P' \;,$$

where $\hat{\mathbf{a}}_\rho$ and $\hat{\mathbf{a}}_\phi$ are base unit vectors at P, and $\hat{\mathbf{a}}_{\rho'}$ and $\hat{\mathbf{a}}_{\phi'}$ are the base vectors at P'. Notice that although **R** is the same vector at P and P', its representations at these two points are different.

2-5 The Calculus of Scalar and Vector Fields

Most of the physical quantities of interest in electromagnetics are field quantities. Because they are functions of position, it is important that we be able to characterize the functional behaviors of field quantities over both large and small regions of space. This is accomplished through various integral and differential operators.

2-5.1 Integrals of Scalar and Vector Fields

Electromagnetic phenomena are often described in terms of integrals of vector or scalar quantities over a volume, a surface, or a line. Examples of the kinds of integrals encountered in electromagnetic analysis are

$$\int_V Q\,dv \qquad 2.69$$

$$\int_V \mathbf{J}\,dv \qquad 2.70$$

$$\int_S \mathbf{D}\bullet\mathbf{ds} \qquad 2.71$$

$$\int_C \mathbf{E}\bullet\mathbf{d}\ell\ . \qquad 2.72$$

The first two integrals above are called ***volume integrals***, because they take place throughout a specified volume. Likewise, the third and fourth integrals are called ***surface*** and ***line*** (or ***contour***) ***integrals*** , respectively, because they are evaluated over a surface or a line, respectively.

In spite of the obvious differences between the integrals given in equations 2.69-2.72, each is simply the summation of a differential quantity (either scalar or vector) over a range of points. The basic steps for evaluating any of these integrals are:

1) Choose the coordinate system that will be used during the integration process.
2) Determine which position coordinates vary during the integration process.
3) Select the appropriate differential quantity.
4) If the integrand is a vector, make sure that all unit vectors are constants with respect to the variable(s) of integration.
5) Integrate over the appropriate limits of the position coordinates.

The three examples that follow demonstrate the general procedure for evaluating integrals of field quantities.

Example 2-6

Evaluate the integral $\int_V \mathbf{P}dv$, where $\mathbf{P} = r\cos\phi\,\hat{\mathbf{a}}_r$ and V is a sphere of unit radius, centered at the origin.

Solution:

The spherical coordinate variables are the most convenient for this problem. To cover all points within this volume, the range of the coordinates must be: $0<r<1$, $0<\theta<\pi$, $0<\phi<2\pi$. Also, $dv = r^2\sin\theta\, dr d\theta d\phi$ at all points within the volume. Substituting into the integral, we have:

$$\int_V \mathbf{P}dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^3\cos\phi\,\sin\theta\,\hat{\mathbf{a}}_r\, dr d\theta d\phi\,.$$

This integral is not as easy to evaluate as it may first appear because of the presence of the unit vector $\hat{\mathbf{a}}_r$, which varies with the position variables θ and ϕ. Using Table B-3, however, we can represent $\hat{\mathbf{a}}_r$ in Cartesian components,

$$\hat{\mathbf{a}}_r = \sin\theta\cos\phi\,\hat{\mathbf{a}}_x + \sin\theta\sin\phi\,\hat{\mathbf{a}}_y + \cos\theta\,\hat{\mathbf{a}}_z\,.$$

Substituting, we obtain

$$\int_V \mathbf{P}dv = \hat{\mathbf{a}}_x\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^3\sin^2\theta\cos^2\phi\, dr d\theta d\phi + \hat{\mathbf{a}}_y\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^3\sin^2\theta\sin\phi\cos\phi\, dr d\theta d\phi$$

$$+\hat{\mathbf{a}}_z\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^3\sin\theta\cos\theta\cos\phi\, dr d\theta d\phi\,.$$

The second and third integrals on the right-hand side of this expression are zero since $\int_0^{2\pi}\sin\phi\cos\phi\, d\phi = 0$ and $\int_0^{2\pi}\cos\phi\, d\phi = 0$, respectively, leaving

$$\int_V \mathbf{P}dv = \hat{\mathbf{a}}_x\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^3\sin^2\theta\cos^2\phi\, dr d\theta d\phi \quad = \frac{\hat{\mathbf{a}}_x}{4}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\cos^2\phi\, d\theta d\phi$$

$$= \frac{\pi\hat{\mathbf{a}}_x}{8}\int_0^{2\pi}\cos^2\phi\, d\phi = \frac{\pi^2}{8}\hat{\mathbf{a}}_x\,.$$

Example 2-7

Evaluate the integral $\oint_S \mathbf{F}\bullet\mathbf{ds}$, where $\mathbf{F} = x\,\hat{\mathbf{a}}_x$ and S is the closed circular cylinder shown in Figure 2-24.

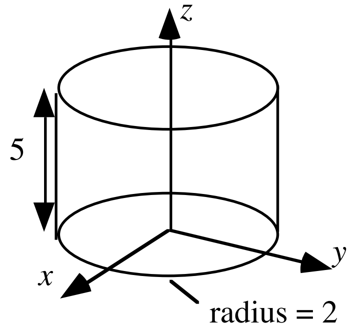


Figure 2-24

A circular cylinder.

Solution:

From Figure 2-24, we see that S consists of two discs at $z= 0$ and $z = 5$, respectively, and an open cylinder $\rho = 2$ for $0 < \phi < 2\pi$. On the discs, $\mathbf{ds} = \pm\rho d\rho d\phi\,\hat{\mathbf{a}}_z$, where the upper and lower signs correspond to the upper and lower discs, respectively. Since $\mathbf{F}$ has no z component, $\mathbf{F}\bullet\mathbf{ds} = 0$ everywhere on both discs.

On the open cylinder, $\mathbf{ds} = \rho d\phi dz\,\hat{\mathbf{a}}_\rho = 2d\phi dz\,\hat{\mathbf{a}}_\rho$. Remembering that $x = \rho\cos\phi$, we can write $\mathbf{F}\bullet\mathbf{ds}$ as

$$\mathbf{F}\bullet\mathbf{ds} = x\,\hat{\mathbf{a}}_x \bullet 2d\phi dz\,\hat{\mathbf{a}}_\rho = 4\cos^2\phi\, d\phi dz \;,$$

where the value of $\hat{\mathbf{a}}_x\bullet\hat{\mathbf{a}}_\rho = \cos\phi$ and $x = \rho\cos\phi$ were obtained from Tables B-2 and B-1, respectively. Substituting into the integral yields

$$\oint_S \mathbf{F}\bullet\mathbf{ds} = \int_0^5 \int_0^{2\pi} 4\cos^2\phi\, d\phi\, dz = \int_0^5 4\pi\, dz = 20\pi \;.$$

Example 2-8

For $\mathbf{F} = y\,\hat{\mathbf{a}}_x - x\,\hat{\mathbf{a}}_y$, evaluate the line integral $\int_C \mathbf{F}\bullet d\ell$ along two paths shown in Figure 2-25, each starting at (0,0,0) and ending at (1,2,4).

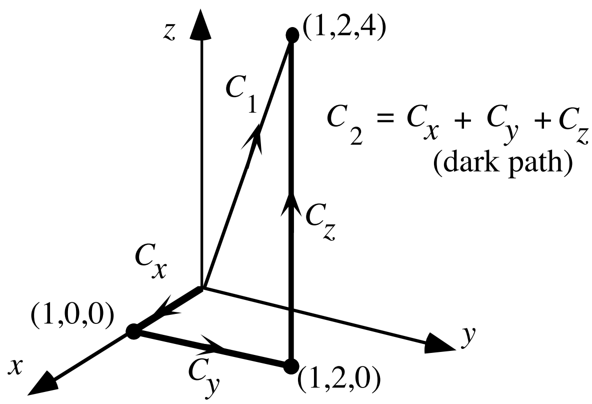


Figure 2-25
Two paths connecting the points (0,0,0) and (1,2,4).

Solution:

For either path, the differential displacement vector can be written in the form $\mathbf{d\ell} = dx\,\hat{\mathbf{a}}_x + dy\,\hat{\mathbf{a}}_y + dz\,\hat{\mathbf{a}}_z$. For the vector **F** given in this problem, the dot product $\mathbf{F}\bullet\mathbf{d\ell}$ is

$$\mathbf{F}\bullet\mathbf{d\ell} = y dx - x dy \quad .$$

a) The path C_1 is a straight line, which can be described by the equations

$$y = 2x$$

$$z = 4x \quad .$$

Since both y and z can be written as functions of x, we can consider x as the parametric variable for use in equations 2.34a through 2.34c. Using these equations, we obtain

$$dx = \frac{\partial x}{\partial x} dx = dx$$

$$dy = \frac{\partial y(x)}{\partial x} dx = 2dx$$

$$dz = \frac{\partial z(x)}{\partial x} dx = 4dx \quad .$$

Using these expressions for dx, dy, and dz, we obtain

$$\mathbf{F}\bullet\mathbf{d\ell} = y dx - x dy = 2x dx - 2x dx = 0 dx \quad .$$

This means that **F** is perpendicular to $\mathbf{d\ell}$ at every point along the path C_1. Integrating, we obtain the result

$$\int_{C_1} \mathbf{F}\bullet\mathbf{d\ell} = \int_0^1 0 dx = 0 \quad .$$

b) Path C_2 is actually a collection of three straight line paths: C_a from (0,0,0) to (1,0,0),

C_b from (1,0,0) to (1,2,0), and C_c from (1,2,0) to (1,2,4) .

$$\int_{C_2} \mathbf{F} \bullet \mathbf{d}\ell = \int_{C_x} \mathbf{F} \bullet \mathbf{d}\ell + \int_{C_y} \mathbf{F} \bullet \mathbf{d}\ell + \int_{C_z} \mathbf{F} \bullet \mathbf{d}\ell .$$

These line integrals are simple to evaluate since, only one position variable varies along each path. Thus, along the paths C_a, C_b and C_c, we have $\mathbf{d}\ell = dx\, \hat{\mathbf{a}}_x$, $\mathbf{d}\ell = dy\, \hat{\mathbf{a}}_y$, and $\mathbf{d}\ell = dz\, \hat{\mathbf{a}}_z$, respectively. Substituting these into the integrals and taking the dot products with **F**, we obtain

$$\int_{C_2} \mathbf{F} \bullet \mathbf{d}\ell = \int_0^1 y\, dx \Big|_{y=0} - \int_0^2 x\, dy \Big|_{x=1} - \int_0^4 0 dz = -2 \quad .$$

Since different answers were obtained when integrating $\mathbf{F} \bullet \mathbf{d}\ell$ along two different paths that connect the same endpoints, **F** is called a ***non-conservative*** vector field. This name comes from mechanics, where, if **F** represents a force, the integral $\int_C \mathbf{F} \bullet \mathbf{d}\ell$ equals the work done on an object as it moves along the path C. For the vector **F** in this problem, the net work done in moving the object from the origin to the point (1,2,4) along C_2 and back to the origin along path C_1 would be -2 - 0 = -2 ≠ 0, which means that the net work done along this closed path is nonzero. Since work is not conserved, the vector **F** is called a ***nonconservative vector.*** On the other hand, vectors for which $\oint_C \mathbf{F} \bullet \mathbf{d}\ell = 0$ for all possible closed paths C are called ***conservative vectors***.

2-5.2 The Gradient of a Scalar Field

Up to this point we have described scalar fields in terms of the rules that determine their values at each point in space. Often, however, the rate at which a scalar changes close to a point is more important than its value at the point itself. When walking up an incline, for example, one is usually more concerned about the change in altitude encountered with each step than with the altitude of each point relative to sea level. The gradient operation provides this kind of information.

To start our discussion, let us consider the change in the value of an arbitrary scalar field f as we move from (x,y,z) to $(x+dx, y+dy, z+dz)$. We will denote this change as df. From ordinary multivariable calculus, df is given by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad . \qquad 2.73$$

This expression can be written as the following dot product between two vectors,

$$df = \left(\frac{\partial f}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial f}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z \right) \bullet \left(dx\, \hat{\mathbf{a}}_x + dy\, \hat{\mathbf{a}}_y + dz\, \hat{\mathbf{a}}_z \right) \qquad 2.74$$

The vector on far right is simply the differential displacement vector $\mathbf{d}\ell$ along the path of movement (see equation 2.64a), so we can write equation 2.74 in the form

$$df = \left(\frac{\partial f}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial f}{\partial y}\hat{\mathbf{a}}_y + \frac{\partial f}{\partial z}\hat{\mathbf{a}}_z \right) \bullet \mathbf{d}\ell \; . \qquad 2.75$$

The vector quantity in parentheses is called the ***gradient*** of f and is denoted symbolically by $\mathbf{grad}\, f$. Hence

$$df = \mathbf{grad}f \bullet \mathbf{d}\ell \; , \qquad 2.76$$

where, in the Cartesian coordinate system,

$$\mathbf{grad}f = \frac{\partial f}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial f}{\partial y}\hat{\mathbf{a}}_y + \frac{\partial f}{\partial z}\hat{\mathbf{a}}_z \; . \quad \text{Cartesian Coordinates} \qquad 2.77$$

We can also write equation 2.77 in the shorthand form,

$$\mathbf{grad}f = \left(\frac{\partial f}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial f}{\partial y}\hat{\mathbf{a}}_y + \frac{\partial f}{\partial z}\hat{\mathbf{a}}_z \right) = \nabla f \; , \qquad 2.78$$

where ∇ is called the ***del operator*** and is defined by

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial}{\partial y}\hat{\mathbf{a}}_y + \frac{\partial}{\partial z}\hat{\mathbf{a}}_z . \quad \text{Cartesian Coordinates} \; . \qquad 2.79$$

Strictly speaking, the del (∇) operator is not a true vector, since its components are operators, rather than numbers. Nevertheless, it is convenient to treat it like a vector in product equations like equation 2.79 and several others in this chapter. Throughout the remainder of this text, we always represent the vector $\mathbf{grad}f$ as ∇f.

Before we derive the representations of ∇f in the other coordinate systems, let us determine general properties of the gradient operation. Using the definition of the dot product (equation 2.8), we can write equation 2.76 in the form

$$df = \nabla f \bullet \mathbf{d}\ell = |\nabla f| d\ell \cos\theta \quad , \qquad 2.80$$

where θ is the angle between ∇f and $\mathbf{d}\ell$. Dividing both sides by $d\ell$ yields

$$\frac{df}{d\ell} = |\nabla f| \cos\theta \quad . \qquad 2.81$$

When the direction of the path is parallel to ∇f, $\cos\theta = 1$. Along such a path, $\frac{df}{d\ell}$ attains its maximum value. Thus,

$$\left.\frac{df}{d\ell}\right|_{max} = |\nabla f| \ . \qquad 2.82$$

Using equation 2.82, we can now define ∇f as

$$\nabla f \equiv \left.\frac{\partial f}{\partial \ell}\right|_{max} \hat{\mathbf{a}}_n \ , \qquad 2.83$$

where $\hat{\mathbf{a}}_n$ points in the direction of maximum increase in f. This definition is valid in all coordinate systems. Thus, the gradient ∇f is a vector that points in the direction of maximum rate of increase of the function f.

From equation 2.80 we see that $df = 0$ whenever $\mathbf{d\ell}$ is perpendicular to ∇f. Thus, ∇f is always perpendicular to surfaces over which f is constant. This can be seen from Figure 2-26, which shows several surfaces of constant value for a scalar function f. As can be seen in this figure, ∇f is perpendicular to each of these surfaces and points towards increasing values of f.

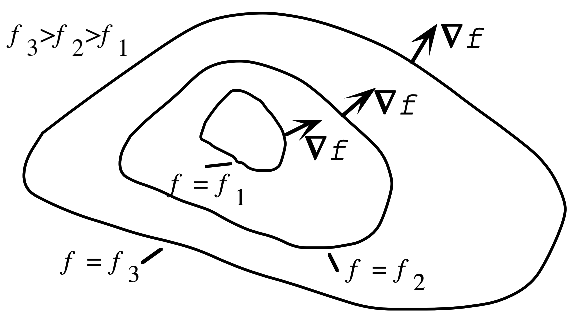


Figure 2-26

Equi-value surfaces and gradient vectors for an arbitrary function f

Representations of ∇f can also be found in the cylindrical and spherical coordinate systems. In cylindrical coordinates, the total differential of a scalar function f is

$$df = \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \phi} d\phi + \frac{\partial f}{\partial z} dz \ ,$$

which can be re-written as the dot product

$$df = \left(\frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z \right) \bullet \left(d\rho\, \hat{\mathbf{a}}_\rho + \rho\, d\phi\, \hat{\mathbf{a}}_\phi + dz\, \hat{\mathbf{a}}_z \right).$$

Since the second vector on the right side of this equation is the differential displacement vector $\mathbf{d\ell}$ in cylindrical coordinates (see equation 2.47), df can be in the form

$$df = \left(\frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z \right) \bullet \mathbf{d\ell} \ .$$

Comparing this to equation 2.80, we see that the vector in the parenthesis must be ∇f. Thus, we have

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\mathbf{a}}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z}\hat{\mathbf{a}}_z . \qquad \text{Cylindrical Coordinates} \qquad 2.84$$

Similarly, in spherical coordinates we can write

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \ ,$$

or

$$df = \left(\frac{\partial f}{\partial r}\hat{\mathbf{a}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{a}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{a}}_\phi\right)\bullet\left(dr\,\hat{\mathbf{a}}_r + r\,d\theta\,\hat{\mathbf{a}}_\theta + r\sin\theta\,d\phi\,\hat{\mathbf{a}}_\phi\right).$$

The second vector on the right side of this expression is $\mathbf{d}\ell$ (see equation 2.61), so this expression can be written as

$$df = \left(\frac{\partial f}{\partial r}\hat{\mathbf{a}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{a}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{a}}_\phi\right)\bullet\mathbf{d}\ell \ .$$

Comparing this with equation 2.80, we can finally write

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{a}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{a}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{a}}_\phi . \qquad \text{Spherical Coordinates} \qquad 2.85$$

Example 2-9

Find the gradient of the scalar field $f = x^2 + y^2$ in a) Cartesian and b) cylindrical coordinates.

Solution:

a) The necessary partial derivatives dictated by equation 2.79 are

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y \qquad \frac{\partial f}{\partial z} = 0 \quad .$$

Therefore,

$$\nabla f = 2x\,\hat{\mathbf{a}}_x + 2y\,\hat{\mathbf{a}}_y \ .$$

b)The representation of ∇f in cylindrical coordinates can be obtained either by transforming the above expression by the normal rules of vector transformations, or by using the cylindrical coordinate expression for ∇f directly. Choosing the latter, we must first express f in cylindrical coordinates:

$$f = x^2 + y^2 = \rho^2\cos^2\phi + \rho^2\sin^2\phi = \rho^2 \ .$$

Next, the necessary partial derivatives of f are

$$\frac{\partial f}{\partial \rho} = 2\rho \qquad \frac{\partial f}{\partial \phi} = 0 \qquad \frac{\partial f}{\partial z} = 0 \ .$$

Substituting these into equation 2.84, we obtain

$$\nabla f = 2\rho\,\hat{\mathbf{a}}_\rho \ .$$

It is left as an exercise for the reader to show that the two expressions for ∇f found in parts *a* and *b* are indeed the same vector.

2-5.3 The Divergence of a vector field

As with scalars, a knowledge of how a vector field changes about a point is often more important than its value at that point. When piloting an airplane, for instance, it is necessary to know whether the air flow at a point is smooth or swirling than it is to know its velocity at a particular point. For vectors, two different indications of their rates of change are necessary to completely characterize these changes. The first of these, called ***divergence***, is discussed in this section. The second, called ***curl***, will be discussed in the section that follows.

The divergence of a vector **A** at a point *P* is as a scalar quantity, defined as

$$\text{div}\,\mathbf{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\bullet\mathbf{ds}}{\Delta v} \ . \qquad 2.86$$

According to this definition, *S* is the surface that bounds the volume Δv, and **ds** always points outward from Δv. The value of surface integral $\oint_S \mathbf{A}\bullet\mathbf{ds}$ indicates whether or not there is a net tendency for **A** to point outward from *P*. Integrals of this type are called ***flux integrals***.

Figures 2-27a&b show two vectors that have nonzero divergence. In the case of Figure 2-27a, the positive divergence of the vector at the origin is easy to see, since all the vector streamlines are directed away from the origin. For this case, $\mathbf{A}\bullet\mathbf{ds}$ is positive at all points on a surface that surrounds the origin, resulting in a net positive flux.

The negative divergence of the vector shown in Figure 2-27b is less obvious to the eye, however, since this vector maintains a general left-to-right direction on both sides of the point P. Nevertheless, the flux entering[7] a surface surrounding P from the left is greater than that which leaves on the right, resulting in a negative divergence at P.

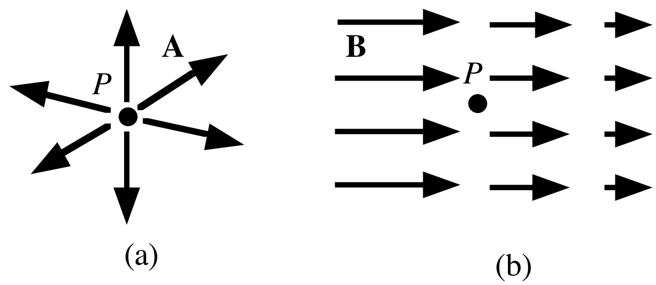


Figure 2-27

Two vector fields that have nonzero divergence at a point P.

Even though the divergence of a vector is defined in terms of a surface integral, we will now show that it can be represented in terms of derivatives of the components of the vector. We will start by evaluating the flux integral $\oint_S \mathbf{A}\bullet\mathbf{ds}$ about the rectangular surface shown in Figure 2-28.

Here, a small rectangular volume of dimensions Δx, Δy, and Δz surrounds the point $P(x_o,y_o,z_o)$, which is shown in the center of the volume. The integral over this closed surface can be written as the sum of six open surface integrals

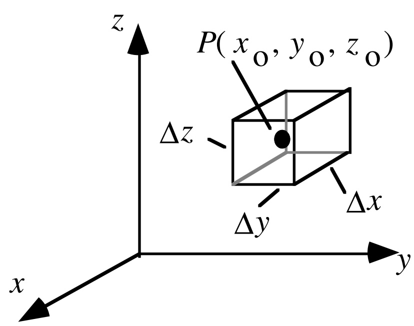


Figure 2-28

A small rectangular surface surrounding a point P.

$$\oint_S \mathbf{A}\bullet\mathbf{ds} = \int_{\substack{\text{front}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{back}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{right}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{left}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{top}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds}. \qquad 2.87$$

On the front face, $x = x_o + \frac{\Delta x}{2}$, $\mathbf{ds} = dydz\, \hat{\mathbf{a}}_\mathbf{x}$, and $\mathbf{A}\bullet\mathbf{ds} = A_x\, dydz$. Substituting, the integral over this face becomes

$$\int_{\substack{\text{front}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} = \int_{y_o-\Delta y/2}^{y_o+\Delta y/2} \int_{z_o-\Delta z/2}^{z_o+\Delta z/2} A_x(x_o+\frac{\Delta x}{2},y,z)\, dy\, dz \ . \qquad 2.88$$

Since Δx, Δy, and Δz are all assumed to be small, we can use Taylor's theorem to expand $A_x(x_o+\frac{\Delta x}{2},y,z)$ about the point $P(x_o,y_o,z_o)$. Using the first two terms of the Taylor's expansion for each coordinate, we obtain:

[7]It is common to speak of flux as if it is something that moves through the surface, even if the vector in question does not represent a quantity of motion (such as a force).

$$A_x\left(x_o+\frac{\Delta x}{2},y,z\right) \cong A_x(x_o,y_o,z_o) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_P + (y-y_o)\frac{\partial A_x}{\partial y}\bigg|_P$$

$$+ (z-z_o)\frac{\partial A_x}{\partial z}\bigg|_P \quad , \qquad 2.89$$

where the notation $\big|_P$ indicates that the derivatives are evaluated at $P(x_o,y_o,z_o)$. Substituting equation 2.89 into 2.88 and integrating, we obtain

$$\int_{\substack{\text{front}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} \cong \Delta y\,\Delta z\,A_x(x_o,y_o,z_o) + \frac{\Delta x}{2}\Delta y\,\Delta z\frac{\partial A_x}{\partial x}\bigg|_P \quad . \qquad 2.90$$

Similarly, on the back face we have $x = x_o - \frac{\Delta x}{2}$, $\mathbf{ds} = -dydz\,\hat{\mathbf{a}}_x$, $\mathbf{A}\bullet\mathbf{ds} = -A_x dydz$, and

$$A_x\left(x_o-\frac{\Delta x}{2},y,z\right) \cong A_x(x_o,y_o,z_o) - \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_P + (y-y_o)\frac{\partial A_x}{\partial y}\bigg|_P + (z-z_o)\frac{\partial A_x}{\partial z}\bigg|_P \quad .$$

Integrating this the back face, we obtain

$$\int_{\substack{\text{back}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} \cong -\Delta y\,\Delta z\,A_x(x_o,y_o,z_o) + \frac{\Delta x}{2}\Delta y\,\Delta z\frac{\partial A_x}{\partial x}\bigg|_P \quad . \qquad 2.91$$

The sum of the flux contributions from the front and back faces is found by adding equations 2.90 and 2.91,

$$\int_{\substack{\text{front}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{back}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} \cong \Delta x\,\Delta y\,\Delta z\frac{\partial A_x}{\partial x}\bigg|_P \quad . \qquad 2.92$$

Using similar steps, it can also be shown that

$$\int_{\substack{\text{right}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{left}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} \cong \Delta x\,\Delta y\,\Delta z\frac{\partial A_y}{\partial y}\bigg|_P \quad . \qquad 2.93$$

and

$$\int_{\substack{\text{top}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} + \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A}\bullet\mathbf{ds} \cong \Delta x\,\Delta y\,\Delta z\frac{\partial A_z}{\partial z}\bigg|_P \quad . \qquad 2.94$$

Summing all the contributions to $\oint_S \mathbf{A}\bullet\mathbf{ds}$ and noting that $\Delta x\,\Delta y\,\Delta z = \Delta v$, we find

$$\oint_S \mathbf{A}\bullet\mathbf{ds} \cong \left\{\frac{\partial A_x}{\partial x}\bigg|_P + \frac{\partial A_y}{\partial y}\bigg|_P + \frac{\partial A_z}{\partial z}\bigg|_P\right\}\Delta v \quad .$$

This expression becomes exact in the limit $\Delta v \to 0$. Comparing this expression with the definition of divergence (equation 2.86), we find

$$\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad , \qquad 2.95$$

where we have dropped the notation $\Big|_P$, since the volume Δv collapses to the point P as $\Delta v \to 0$. This equation can also be written as a dot product,

$$\text{div } \mathbf{A} = \left(\frac{\partial}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial}{\partial z} \hat{\mathbf{a}}_z \right) \bullet (A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z) = \nabla \bullet \mathbf{A} \ ,$$

where ∇ is the del operator, defined by equation 2.79. Thus, the notation $\nabla \bullet \mathbf{A}$ has the same meaning as div **A**. In the Cartesian coordinate system, can write

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{Cartesian Coordinates} \qquad 2.96$$

Expressions for $\nabla \bullet \mathbf{A}$ also can also be derived in the cylindrical and spherical coordinate systems. This can be accomplished in either of two ways. The first is to transform the Cartesian coordinate expression into these coordinate systems by the standard transformation rules outlined in section 2-4.4. This procedure is straightforward but tedious, since the chain rule must be used repeatedly to transform the variables in the partial derivatives. The second method is to evaluate $\oint_S \mathbf{A} \bullet \mathbf{ds}$ directly in the cylindrical and spherical coordinates systems using a procedure similar to what we used in Cartesian coordinates[8]. By either technique, it can be shown that

$$\nabla \bullet \mathbf{A} = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\rho) \right] + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad \text{Cylindrical Coordinates} \qquad 2.97$$

and

$$\nabla \bullet \mathbf{A} = \frac{1}{r^2} \left[\frac{\partial}{\partial r} (r^2 A_r) \right] + \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (A_\theta \sin\theta) \right] + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} . \quad \text{Spherical Coordinates} \qquad 2.98$$

[8] See Plonsey and Collin, *Principles and Applications of Electromagnetic Fields*, McGraw-Hill, New York 1961

Example 2-10

Find the divergence of $\mathbf{A} = x\,\hat{\mathbf{a}}_x$ at any point using a) Cartesian coordinates and b) Cylindrical coordinates.

Solution:

a) From equation 2.96,

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial x}(x) = 1 \quad \text{(at all points)} \ .$$

b) Transforming **A** into cylindrical coordinates, we find that

$$\mathbf{A} = \rho \cos^2\phi \, \hat{\mathbf{a}}_\rho - \rho \cos\phi \sin\phi \, \hat{\mathbf{a}}_\phi \ .$$

Using equation 2.97,

$$\nabla \bullet \mathbf{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho^2 \cos^2\phi)\right] - \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho \cos\phi \sin\phi)$$

$$= 2\cos^2\phi - (-\sin^2\phi + \cos^2\phi) = 1 \ .$$

As expected, this result is the same as was obtained using the Cartesian coordinate system.

Before leaving the subject of divergence, we will derive an important theorem involving divergence, called the ***divergence theorem***. Consider the volume integral $\int_V \nabla \bullet \mathbf{A}\, dv$. Using the definition of divergence, we can write this integral as

$$\int_V \nabla \bullet \mathbf{A}\, dv = \int_V \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \bullet d\mathbf{s}}{\Delta v}\, dv \ .$$

Expressing the right hand integral as an infinite sum of infinitesimal volumes, we obtain

$$\int_V \nabla \bullet \mathbf{A}\, dv = \sum_k \lim_{\Delta v_k \to 0} \frac{\oint_{S_k} \mathbf{A} \bullet d\mathbf{s}}{\Delta v_k} \Delta v_k \ ,$$

where Δv_k is the k^{th} differential subvolume, which is surrounded by the closed surface S_k .

The right hand side of the above expression allows us to interpret $\int_V \nabla\bullet\mathbf{A}dv$ as the sum of the fluxes emanating from each point within V.

But as can be seen from Figure 2-29, flux contributions from adjacent points within V cancel, since the outward flux from one volume is at the same time inward flux to its neighbor. All flux contributions in the integral cancel, except those at points on the surface bounding V. Thus,

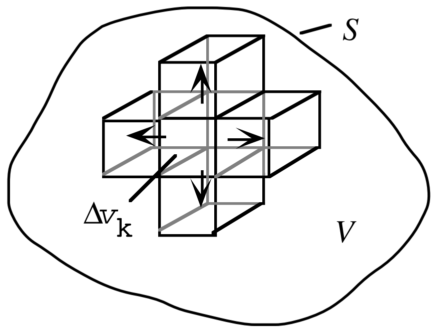


Figure 2-29

Geometry for deriving the divergence theorem.

$$\int_V \nabla\bullet\mathbf{A}dv = \sum_k \lim_{\Delta v_k \to 0} \oint_{S_k} \mathbf{A}\bullet\mathbf{ds} = \oint_S \mathbf{A}\bullet\mathbf{ds} \ .$$

The sum on the right-hand side is the integral $\oint_S \mathbf{A}\bullet\mathbf{ds}$, where S is the closed surface that bounds the volume V. Hence, we obtain the ***divergence theorem***,

$$\int_V \nabla\bullet\mathbf{A}dv = \oint_S \mathbf{A}\bullet\mathbf{ds} \ . \qquad \text{Divergence Theorem} \qquad 2.99$$

This theorem is useful for transforming equations involving vector integrals into simpler forms.

Example 2-11

Given $\mathbf{A} = r\,\hat{\mathbf{a}}_r + \sin\theta\ \hat{\mathbf{a}}_\theta$, verify the divergence theorem over the spherical volume of radius $r = 1$, centered about the origin.

Solution:

For the surface integral,

$$\mathbf{ds} = \mathbf{ds}_r = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{a}}_r$$

and

$$\mathbf{A}\bullet\mathbf{ds} = r^3 \sin\theta\, d\theta\, d\phi\, .$$

Substituting, the surface integral becomes

$$\oint_S \mathbf{A}\bullet\mathbf{ds} = \int_0^{2\pi}\int_0^{\pi} r^3 \sin\theta\, d\theta\, d\phi \Big|_{r=1} = 2\pi \int_0^{\pi} \sin\theta\, d\theta = 4\pi\, .$$

To evaluate the volume integral, we must first evaluate the divergence of **A**,

$$\nabla \bullet \mathbf{A} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^3)\right] + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin^2\theta)\right]$$

$$= 3 + \frac{2\cos\theta}{r} .$$

Substituting, the volume integral becomes

$$\int_V \nabla \bullet \mathbf{A} dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\left(3 + \frac{2\cos\theta}{r}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 3 r^3 \sin\theta \, dr d\theta d\phi + \int_0^{\pi}\int_0^{2\pi}\int_0^{1} 2r \sin\theta \cos\theta \, dr d\theta d\phi$$

$$= 4\pi + 0 = 4\pi .$$

The two integrals are indeed equal, just as predicted by the divergence theorem.

2-5.4 The Curl of a Vector Field

The ***curl*** of a vector **A** is an indication of its tendency to "push" or "pull" along a closed path that encircles a point. By this we mean that a vector **A** has curl at a point if the line integral $\oint_C \mathbf{A} \bullet d\ell$ is nonzero and C is a differential path that encircles the point. This tendency to push or pull around a path is called ***circulation***. There are three perpendicular planes that such a path can lie in about a point, so the curl is defined as a vector quantity, denoted by the symbols "curl **A**" or "$\nabla \times \mathbf{A}$". Referring to Figure 2-30, we define the component of $\nabla \times \mathbf{A}$ in the direction $\hat{\mathbf{a}}_i$ by

$$(\text{curl } \mathbf{A})_i \equiv (\nabla \times \mathbf{A})_i \equiv \lim_{\Delta s_i \to 0} \frac{\oint_{C_i} \mathbf{A} \bullet d\ell}{\Delta s_i} , \qquad 2.100$$

where Δs_i is a small surface that is bounded by the contour (i.e., path) C_i and has unit normal $\hat{\mathbf{a}}_i$. The direction of C_i is governed by the right-hand rule, which says that when the right-hand thumb is placed along the path, the remaining fingers "poke" through the surface Δs_i in the direction of $\hat{\mathbf{a}}_i$.

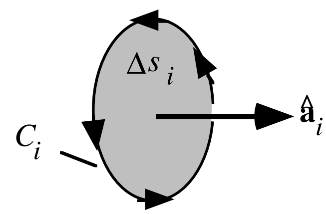


Figure 2-30
A surface Δs_i with unit normal $\hat{\mathbf{a}}_i$, bounded by the contour C_i

Since $\nabla \times \mathbf{A}$ is a vector, we can represent it by its magnitude and direction, which we will denote as $|\nabla \times \mathbf{A}|$ and $\hat{\mathbf{a}}_n$, respectively. To find $|\nabla \times \mathbf{A}|$, we notice from equation 2.100 that the values of the components of $\nabla \times \mathbf{A}$ vary with the orientations of the integration paths C_i . Since the maximum value that any component of a vector can attain equals the vector's magnitude, we can conclude that

$$|\nabla \times \mathbf{A}| = \left[\lim_{\Delta s \to 0} \frac{\oint_C \mathbf{A} \bullet d\ell}{\Delta s} \right]_{max} , \qquad 2.101$$

where C is the differential path that maximizes the circulation integral. Thus, we can write **curl A** as

$$\nabla \times \mathbf{A} \equiv \hat{\mathbf{a}}_n \left[\lim_{\Delta s \to 0} \frac{\oint_C \mathbf{A} \bullet d\ell}{\Delta s} \right]_{max} , \qquad 2.102$$

where $\hat{\mathbf{a}}_n$ is perpendicular to the surface bounded by C and points in the direction determined by the right-hand rule.

Figure 2-31 is helpful in understanding the meaning of the vector $\nabla\times\mathbf{A}$. Here, a paddle wheel is placed in a fluid whose velocity is represented by the vector **A**. A torque will be exerted on the paddle wheel whenever there is a nonzero circulation of **A** about the paddle wheel axis. According to equation 2.102, maximum torque is produced when the axis of the wheel is in the direction of $\nabla\times\mathbf{A}$. If no torque is produced at a point for any orientation of the wheel, **A** has no curl there.

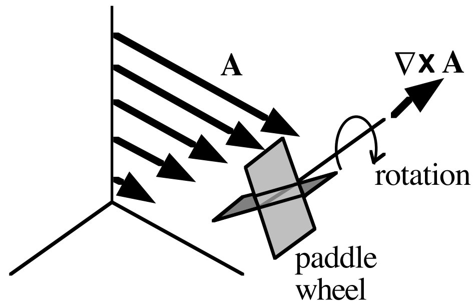


Figure 2-31
"Paddle wheel" analogy of the curl of a vector.

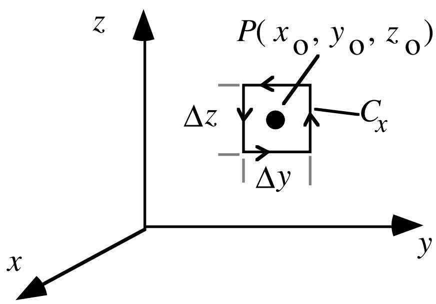


Figure 2-32
A contour C_x in the *yz* plane, about an arbitrary point *P*

The curl of a vector can be calculated by evaluating partial derivatives of the components of **A** with respect to the coordinate variables. To show this, let us first find the *x* component of $\nabla\times\mathbf{A}$, which requires that we evaluate $\oint_{C_x}\mathbf{A}\bullet d\ell$ along the contour C_x, shown in Figure 2-32. This integral can be written as the sum of four line integrals

$$\oint_{C_x}\mathbf{A}\bullet\mathbf{d}\ell = \int_{\text{right}}\mathbf{A}\bullet\mathbf{d}\ell + \int_{\text{top}}\mathbf{A}\bullet\mathbf{d}\ell + \int_{\text{left}}\mathbf{A}\bullet\mathbf{d}\ell + \int_{\text{bottom}}\mathbf{A}\bullet\mathbf{d}\ell . \qquad 2.103$$

Along the right and left contours, $\mathbf{d}\ell = dz\,\hat{\mathbf{a}}_z$. Similarly, $\mathbf{d}\ell = dy\,\hat{\mathbf{a}}_y$ along the top and bottom contours. Substituting these into the contour integrals, we find

$$\oint_{C_x}\mathbf{A}\bullet\mathbf{d}\ell = \int_{z_o-\frac{\Delta z}{2}}^{z_o+\frac{\Delta z}{2}} A_z(x_o,y_o+\frac{\Delta y}{2},z)\,dz + \int_{y_o+\frac{\Delta y}{2}}^{y_o-\frac{\Delta y}{2}} A_y(x_o,y,z_o+\frac{\Delta z}{2})\,dy$$

$$+ \int_{z_o+\frac{\Delta z}{2}}^{z_o-\frac{\Delta z}{2}} A_z(x_o,y_o-\frac{\Delta y}{2},z)\,dz + \int_{y_o-\frac{\Delta y}{2}}^{y_o+\frac{\Delta y}{2}} A_y(x_o,y,z_o-\frac{\Delta z}{2})\,dy . \qquad 2.104$$

Note that the limits of integration are such that the path of integration is counter-clockwise, which is consistent with the right-hand rule.

Since both Δx and Δy are small, each of the integrands on the right-hand side of equation 2.104 can be expanded in a Taylor's series about $P(x_o,y_o,z_o)$. For the first integral, we can write

$$A_z(x_o,y_o+\frac{\Delta y}{2},z) \cong A_z(x_o,y_o,z_o) + \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\Big|_P + (z - z_o)\frac{\partial A_z}{\partial z}\Big|_P . \qquad 2.105$$

Integrating, we obtain

$$\int_{\text{right}} \mathbf{A}\bullet\mathbf{d}\ell \cong \Delta z\, A_z\,(x_o,y_o,z_o) + \frac{\Delta y}{2}\,\Delta z\,\frac{\partial A_z}{\partial y}\Big|_P . \qquad 2.106$$

Similarly, for integral over the left segment of C_x , we can express the integrand as

$$A_z(x_o,y_o-\frac{\Delta y}{2},z) \cong A_z(x_o,y_o,z_o) - \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\Big|_P + (z-z_o)\frac{\partial A_z}{\partial z}\Big|_P , \qquad 2.107$$

which yields

$$\int_{\text{left}} \mathbf{A}\bullet\mathbf{d}\ell \cong -\Delta z\, A_z\,(x_o,y_o,z_o) + \frac{\Delta y}{2}\,\Delta z\,\frac{\partial A_z}{\partial y}\Big|_P . \qquad 2.108$$

Summing the contributions of from the "right" and "left" portions of the contour yields

$$\int_{\text{right}} \mathbf{A}\bullet\mathbf{d}\ell + \int_{\text{left}} \mathbf{A}\bullet\mathbf{d}\ell \cong \Delta y\, \Delta z\,\frac{\partial A_z}{\partial y}\Big|_P . \qquad 2.109$$

Similar analysis of the "top" and "bottom" portions of the contour yields

$$\int_{\text{top}} \mathbf{A}\bullet\mathbf{d}\ell + \int_{\text{bottom}} \mathbf{A}\bullet\mathbf{d}\ell \cong -\Delta y\, \Delta z\,\frac{\partial A_y}{\partial z}\Big|_P . \qquad 2.110$$

Substituting equations 2.109 and 2.110 into equation 2.103, we have

$$\oint_{C_x} \mathbf{A}\bullet\mathbf{d}\ell \cong \Delta y\, \Delta z \left[\frac{\partial A_z}{\partial y}\Big|_P - \frac{\partial A_y}{\partial z}\Big|_P \right] . \qquad 2.111$$

which becomes exact in the limit as $\Delta s_x = \Delta y\, \Delta z \rightarrow 0$. Comparing this expression with equation 2.100, we can conclude

$$(\nabla \mathbf{x} \mathbf{A})_x = \lim_{\Delta s_x \rightarrow 0} \frac{\oint_{C_x} \mathbf{A}\bullet\mathbf{d}\ell}{\Delta s_x} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} , \qquad 2.112$$

where the notation $\Big|_P$ has been dropped from the partial derivatives since the surface has collapsed to a point.

The y and z components of **∇xA** can be found by evaluating equation 2.100 around the contours C_y and C_z, which lie in the $y=y_o$ and $z=z_o$ planes, respectively. Evaluating the resulting circulation integrals using the same procedure as used for $(\nabla \mathbf{x} \mathbf{A})_x$, we finally obtain

$$\nabla \mathbf{x} \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{\mathbf{a}}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{\mathbf{a}}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{\mathbf{a}}_z$$

Cartesian Coordinates 2.113

This formula can also be written in shorthand form as a determinant,

$$\nabla \mathbf{x} \mathbf{A} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} , \qquad 2.114$$

which shows why the symbol "**∇xA**" and "curl **A**" are used interchangeably..

Corresponding expressions for **∇xA** exist in the cylindrical and spherical coordinate systems

$$\nabla \mathbf{x} \mathbf{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\mathbf{a}}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{\mathbf{a}}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right]\hat{\mathbf{a}}_z$$

Cylindrical Coordinates 2.115

and

$$\nabla \mathbf{x} \mathbf{A} = \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right]\hat{\mathbf{a}}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\mathbf{a}}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\mathbf{a}}_\phi .$$

Spherical Coordinates 2.116

These expressions can be derived either by transforming the components and coordinates of equation 2.113 into the new coordinate system, or by evaluating the circulation integrals of equation 2.100 directly in the new coordinate systems[9].

Example 2-12

Calculate the curl of $\mathbf{A} = y\,\hat{\mathbf{a}}_x$ at all points using a) Cartesian coordinates and b) spherical coordinates.

Solution:

a) Of the six partial derivatives present in the expression for $\nabla \times \mathbf{A}$, only one is nonzero, since $A_y = A_z = 0$ and A_x is a function only of y. Thus,

$$\nabla \times \mathbf{A} = -\frac{\partial A_x}{\partial y}\,\hat{\mathbf{a}}_z = -\hat{\mathbf{a}}_z\ .$$

b) Transforming $\mathbf{A}$ into spherical coordinates, we find

$$\mathbf{A} = r\sin^2\theta\sin\phi\cos\phi\,\hat{\mathbf{a}}_r + r\sin\theta\cos\theta\sin\phi\cos\phi\,\hat{\mathbf{a}}_\theta - r\sin\theta\sin\phi\,\hat{\mathbf{a}}_\phi\ .$$

From equation 2.116, we have

$$(\nabla \times \mathbf{A})_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}[\sin\theta\,(-r\sin^2\theta\sin^2\phi)]\right.$$

$$\left. -\frac{\partial}{\partial\phi}(r\sin\theta\cos\theta\sin\phi\cos\phi)\right) = -\cos\theta$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\sin^2\theta\sin\phi\cos\phi) - \frac{\partial}{\partial r}(-r^2\sin\theta\sin^2\phi)\right] = \sin\theta$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cos\theta\sin\phi\cos\phi) - \frac{\partial}{\partial\theta}(r\sin^2\sin\phi\cos\phi)\right] = 0\ .$$

Thus, $\nabla \times \mathbf{A} = -\cos\theta\,\hat{\mathbf{a}}_r + \sin\theta\,\hat{\mathbf{a}}_\theta$. It is left as an exercise to the reader to show that this result is equivalent to the one found in part a).

---

[9] See Plonsey and Collin, *Principles and Applications of Electromagnetic Fields*, McGraw-Hill, New York, 1961

A useful theorem that involves the curl operation is ***Stokes Theorem***. To derive this theorem, consider the integral $\int_S (\nabla \times \mathbf{A}) \bullet \mathbf{ds}$ over an open surface S. From the properties of the dot product, we can write the integrand as

$$(\nabla \times \mathbf{A}) \bullet \mathbf{ds} = (\nabla \times \mathbf{A}) \bullet ds\, \hat{\mathbf{a}}_n = (\nabla \times \mathbf{A})_n \, ds \ , \qquad 2.117$$

where $\hat{\mathbf{a}}_n$ is the outward normal to the differential surface and $(\nabla \times \mathbf{A})_n$ is the component of $\nabla \times \mathbf{A}$ in the $\hat{\mathbf{a}}_n$ direction. Using this, we can write $\int_S (\nabla \times \mathbf{A}) \bullet \mathbf{ds}$ in the form

$$\int_S (\nabla \times \mathbf{A}) \bullet \mathbf{ds} = \int_S (\nabla \times \mathbf{A})_n \, ds \ .$$

Substituting equation 2.100 into the right-hand side of this expression, we obtain

$$\int_S (\nabla \times \mathbf{A}) \bullet \mathbf{ds} = \int_S \lim_{\Delta s \to 0} \left[\frac{\oint_{\Delta C} \mathbf{A} \bullet \mathbf{d}\ell}{\Delta s} \right] ds \ ,$$

where the contour ΔC bounds the surface $\mathbf{\Delta s} = \Delta s \hat{\mathbf{a}}_n$ in a right-handed sense. Expressing the right-hand integral as an infinite sum of differential surface elements, we obtain

$$\int_S (\nabla \times \mathbf{A}) \bullet \mathbf{ds} = \sum_k \lim_{\Delta s_k \to 0} \frac{\oint_{C_k} \mathbf{A} \bullet \mathbf{d}\ell}{\Delta s_k} \Delta s_k \ .$$

Canceling the Δs_k terms, we find

$$\int_S (\nabla \times \mathbf{A}) \bullet \mathbf{ds} = \sum_k \lim_{\Delta s_k \to 0} \oint_{C_k} \mathbf{A} \bullet \mathbf{d}\ell \ .$$

As can be seen from Figure 2-33, the line integral contributions from adjacent cells cancel, since the directions of integration along these paths are opposite. As a result, all line integral contributions cancel, except those along the contour that bounds S. Thus,

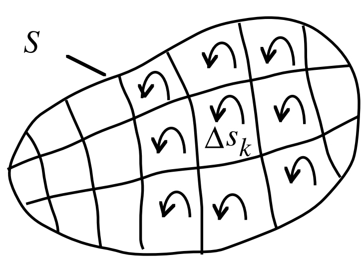


Figure 2-33
Geometry for deriving Stoke's theorem.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{ds} = \oint_C \mathbf{A} \cdot \mathbf{d}\ell \quad \text{Stokes Theorem ,} \qquad 2.118$$

where C is the contour that bounds S in a right-handed sense. If S is a closed surface, it has no bounding contour, so

$$\oint_S \nabla \times \mathbf{A} \cdot \mathbf{ds} = 0 \ . \qquad 2.119$$

An important consequence of Stokes Theorem is that there is more than one surface that corresponds to a particular contour C. This is depicted in Figure 2-34, where the surfaces S_1 and S_2 both have the same bounding contour C. Since both surfaces are bounded by the closed contour C, it follows from Stoke's theorem that both surface integrals have the same value

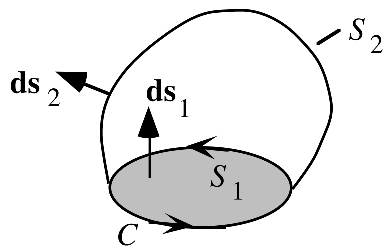


Figure 2-34
Two surfaces bounded by the same contour C.

$$\int_{S_1} (\nabla \times \mathbf{A}) \cdot \mathbf{ds}_1 = \int_{S_2} (\nabla \times \mathbf{A}) \cdot \mathbf{ds}_2 \ . \qquad 2.120$$

The appropriate orientation of the differential surface vector $\mathbf{ds}_1$ on S_1 is easy to visualize from the right-hand rule, since S_1 is flat. Because S_2 is curved, however, the correct orientation of $\mathbf{ds}_2$ is not as obvious. An aid that is helpful here is to imagine that S_1 is an elastic membrane that, when stretched, assumes the shape of S_2. During this process, we simply allow **ds** at each point to remain perpendicular to the surface as the membrane transforms from S_1 to S_2.

Example 2-13

For $\mathbf{A} = \rho z\, \hat{\mathbf{a}}_\phi$, evaluate both sides of Stoke's theorem for the contour C and the surfaces S_1 and S_2, shown in Figure 2-35.

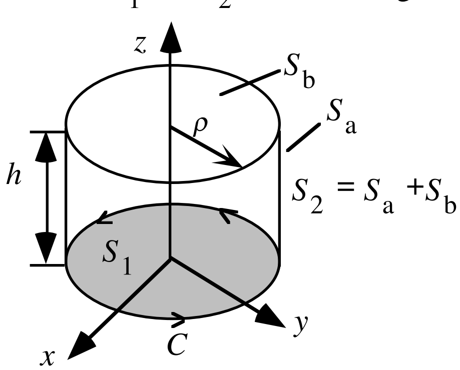


Figure 2-35
Circular contour C that bounds two open surfaces: S_1 and S_2.

Solution:

Both S_1 and S_2 are bounded by the contour C, which is described by $\rho = 2$, $0 < \phi < 2\pi$, and $z = 0$. Along this contour,

$$\mathbf{d}\ell = \rho\, d\phi\, \hat{\mathbf{a}}_\phi \Big|_{\rho=2} = 2\, d\phi\, \hat{\mathbf{a}}_\phi .$$

Substituting this $\mathbf{d}\ell$ into the contour integral, we obtain

$$\oint_C \mathbf{A} \bullet \mathbf{d}\ell = \oint_C 2\rho z\, \hat{\mathbf{a}}_\phi \bullet \hat{\mathbf{a}}_\phi d\phi \Big|_{\substack{\rho=2 \\ z=0}} = \int_0^{2\pi} 0\, d\phi = 0 .$$

To evaluate the surface integrals, we must first calculate $\nabla \times \mathbf{A}$. Since $\mathbf{A}$ has only a ϕ component, we have

$$\nabla \times \mathbf{A} = -\frac{\partial A_\phi}{\partial z}\hat{\mathbf{a}}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\,\hat{\mathbf{a}}_z = -\rho\,\hat{\mathbf{a}}_\rho + 2z\,\hat{\mathbf{a}}_z .$$

For $\int_{S_1} (\nabla \times \mathbf{A}) \bullet \mathbf{ds}$, we note that S_1 is a circle of radius $\rho = 2$. Since the direction of C is counterclockwise, the right-hand rule requires that $\mathbf{ds} = \rho\, d\rho\, d\phi \hat{\mathbf{a}}_z$. But, since the surface is in the z=0 plane, $(\nabla \times \mathbf{A}) \bullet \mathbf{ds} \Big|_{z=0} = 0$, yielding

$$\int_{S_1} (\nabla \times \mathbf{A}) \bullet \mathbf{ds} = \int_{S_1} 0\, d\rho\, d\phi = 0 .$$

The surface S_2 consists of two simple surfaces - a cylinder S_a and its endcap S_b. From the right-hand rule, the differential surface vectors on S_a and S_b are $\mathbf{ds}_a = \rho\, d\phi\, dz\, \hat{\mathbf{a}}_\rho$ and $\mathbf{ds}_b = \rho\, d\rho\, d\phi\, \hat{\mathbf{a}}_z$, respectively. Given these, the surface integral over S_2 becomes:

$$\int_{S_2} (\nabla \times \mathbf{A}) \bullet \mathbf{ds} = \int_0^h \int_0^{2\pi} -\rho^2 d\phi\, dz \Big|_{\rho=2} + \int_0^{2\pi} \int_0^2 2z\rho\, d\rho\, d\phi \Big|_{z=h} = -8\pi h + 8\pi h = 0 .$$

2-5.5 The Laplacian Operator

There are many occasions in vector analysis where a gradient operation is followed by a divergence operation. This combined operation is called the Laplacian and is denoted by the symbol ∇^2,

$$\nabla^2 \equiv \nabla \bullet \nabla \quad . \qquad 2.121$$

The application of the Laplacian to scalar fields is straightforward. In Cartesian coordinates, we have

$$\nabla^2 f = \nabla \bullet \nabla f = \nabla \bullet \left(\frac{\partial f}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial f}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z \right) \ , \qquad 2.122$$

which yields

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{Cartesian Coordinates} \ . \qquad 2.123$$

Similarly, the Laplacian of a scalar field can be expanded in cylindrical and spherical coordinates to yield:

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \ , \quad \text{Cylindrical Coordinates} \qquad 2.124$$

and

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2} \ .$$

Spherical Coordinates 2.125

The Laplacian operator can also be applied to vector fields. To see how this is possible, let us consider the Laplacian of a vector **A** that is represented in Cartesian coordinates,

$$\nabla^2 \mathbf{A} = \nabla^2 \left(A_x \hat{\mathbf{a}}_x + A_y \hat{\mathbf{a}}_y + A_z \hat{\mathbf{a}}_z \right) \ .$$

Since the unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, and $\hat{\mathbf{a}}_z$ are not functions of position, they are constants with respect to the ∇^2 operator. Thus, we can conclude that the Laplacian of a vector is also a vector, with Cartesian components given by

$$\nabla^2\mathbf{A} \equiv \hat{\mathbf{a}}_x\nabla^2 A_x + \hat{\mathbf{a}}_y\nabla^2 A_y + \hat{\mathbf{a}}_z\nabla^2 A_z \quad \text{. Cartesian Components} \qquad 2.126$$

If a vector is expressed in non-Cartesian components, its Laplacian cannot be evaluated so simply. To derive a general expression for $\nabla^2\mathbf{A}$, we note that the right hand side of equation 2.126 can be written as

$$\nabla^2\mathbf{A} = \hat{\mathbf{a}}_x\nabla^2 A_x + \hat{\mathbf{a}}_y\nabla^2 A_y + \hat{\mathbf{a}}_z\nabla^2 A_z = \nabla(\nabla\bullet\mathbf{A}) - \nabla\mathbf{x}\nabla\mathbf{x}\mathbf{A} \ . \qquad 2.127$$

The proof of this identity is straightforward in Cartesian coordinates and is left as an exercise for the reader. Since the divergence and curl operations are well defined in all coordinate systems, the right side of equation 2.127 can be evaluated in any coordinate system. Thus, the Laplacian of a vector can be expressed in all coordinate systems as

$$\nabla^2\mathbf{A} \equiv \nabla(\nabla\bullet\mathbf{A}) - \nabla\mathbf{x}\nabla\mathbf{x}\mathbf{A} \ . \qquad 2.128$$

2-5.6 Helmholtz's Theorem

An important question in vector analysis is, "What kind of information is necessary to completely characterize a vector field over some region of space?". The answer to this question is important for two reasons. First, it allows us to judge whether a particular set of specifications uniquely defines a vector within some region. Second, a knowledge of the minimum information necessary to uniquely specify a vector quantity can simplify the work necessary to solve a given problem.

The key to determining the behavior of any quantity over a region is knowing how it changes from point to point. For scalars, the gradient operation supplies all of the information necessary. The following theorems make it clear that for vectors, two operations are needed: divergence and curl.

Theorem I: Any vector field that is continuously differentiable in some volume V can be uniquely determined if its divergence and curl are known throughout the volume and its value is known on the surface S that bounds the volume:

$$\mathbf{A}(\mathbf{r}) = -\nabla\left[\int_V \frac{\nabla'\bullet\mathbf{A}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}dv' - \oint_S \frac{\mathbf{A}(\mathbf{r}')\bullet\hat{\mathbf{a}}_{n'}}{4\pi|\mathbf{r}-\mathbf{r}'|}ds'\right]$$

$$+ \nabla\mathbf{x}\left[\int_V \frac{\nabla'\mathbf{x}\mathbf{A}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}dv' - \oint_S \frac{\mathbf{A}(\mathbf{r}')\mathbf{x}\hat{\mathbf{a}}_{n'}}{4\pi|\mathbf{r}-\mathbf{r}'|}ds'\right] . \qquad 2.129$$

In this expression, the unit vector $\hat{\mathbf{a}}_{n'}$ points outward from S. Also, inside the integrals, the dummy integration position variable is

$\mathbf{r}' = x'\hat{\mathbf{a}}_x + y'\hat{\mathbf{a}}_y + z'\hat{\mathbf{a}}_z$ and the del operator ∇' is given by $\nabla' = \frac{\partial}{\partial x'}\hat{\mathbf{a}}_x + \frac{\partial}{\partial y'}\hat{\mathbf{a}}_y + \frac{\partial}{\partial z'}\hat{\mathbf{a}}_z$. This is relationship is called ***Helmholtz's Theorem*** and is proved in a number of advanced electromagnetics and mathematics texts[10].

For most vectors found in electromagnetics, the surface integrals in this expression vanish when the volume V is chosen to be all of space. This means that these vectors can be uniquely specified when their curl and divergences are known at all points in space.

Theorem II: Any vector field that is continuously differentiable in some region can be expressed at every point in this volume as the sum of an irrotational vector and a solenoidal vector. Thus,

$$\mathbf{A} = \nabla f + \nabla \times \mathbf{G} \ , \qquad 2.130$$

where f is a scalar field and **G** is a vector field. This identity follows directly from Helmholtz's theorem.

Theorem III: If $\nabla \times \mathbf{A} = 0$ throughout a region, then **A** can be represented as

$$\mathbf{A} = \nabla f \qquad 2.131$$

throughout this region, where f is a scalar field. Vectors for which $\nabla \times \mathbf{A} = 0$ are called ***irrotational vectors***. This theorem follows from Helmholtz's theorem and the identity $\nabla \times \nabla f = 0$ (equation B.9).

Theorem IV: If $\nabla \bullet \mathbf{A} = 0$ throughout a region, then **A** can be represented as

$$\mathbf{A} = \nabla \times \mathbf{G} \qquad 2.132$$

throughout this region, where **G** is a vector field. Vectors for which $\nabla \bullet \mathbf{A} = 0$ are called ***solenoidal vectors***. This theorem follows from Helmholtz's theorem and the identity $\nabla \bullet \nabla \times \mathbf{G} = 0$ (equation B.8).

2-6 Summation

In this chapter, we have presented the basic concepts of vector analysis. While these concepts are firmly rooted in mathematics, our interest in them is solely in their ability to describe physical processes that involve scalar and vector quantities. In the chapters to follow, we will use these concepts freely as we develop the basic equations that define electromagnetics. Vector analysis will also form the basic framework of our analysis and design of electromagnetic systems.

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[10] For instance, see *Principles and Applications of Electromagnetic Fields*, by Robert Plonsey and Robert Collin, McGraw-Hill, New York, 1961

Problems

2-1 If $\mathbf{A} = 3\hat{\mathbf{a}}_x + 2\hat{\mathbf{a}}_y - 4\hat{\mathbf{a}}_z$ and $\mathbf{B} = -2\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y + 2\hat{\mathbf{a}}_z$, find:

a) $|\mathbf{A}|$
b) $|\mathbf{B}|$
c) $\hat{\mathbf{a}}_B$
d) $\mathbf{A}+\mathbf{B}$
e) $\mathbf{A}\bullet\mathbf{B}$
f) The minimum angle θ between **A** and **B**.

2-2 If $\mathbf{A} = -\hat{\mathbf{a}}_x + 3\hat{\mathbf{a}}_y - 2\hat{\mathbf{a}}_z$ and $\mathbf{B} = 2\hat{\mathbf{a}}_x + 3\hat{\mathbf{a}}_y - 2\hat{\mathbf{a}}_z$, find:

a) $|\mathbf{A}|$
b) $|\mathbf{B}|$
c) $\mathbf{A}-\mathbf{B}$
d) $\mathbf{A}\times\mathbf{B}$
e) The minimum angle θ between **A** and **B**.

2-3 If $\mathbf{A} = 2\hat{\mathbf{a}}_\rho - \hat{\mathbf{a}}_\phi - 2\hat{\mathbf{a}}_z$, $\mathbf{B} = 3\hat{\mathbf{a}}_\rho + 2\hat{\mathbf{a}}_\phi + 4\hat{\mathbf{a}}_z$, and $\mathbf{C} = \hat{\mathbf{a}}_\rho + 2\hat{\mathbf{a}}_\phi + \hat{\mathbf{a}}_z$, find:

a) $\mathbf{A}\bullet\mathbf{B}$
b) minimum angle θ_{AB} between **A** and **B**
c) $\mathbf{A}\times\mathbf{B}$
d) the unit vector $\hat{\mathbf{a}}_n$ that points in the direction of $\mathbf{A}\times\mathbf{B}$
e) $\mathbf{C}\bullet\mathbf{A}\times\mathbf{B}$
f) $\mathbf{C}\times(\mathbf{A}\times\mathbf{B})$

2-4 Using the Cartestin coordinate system, prove that the following properties of vector addition are true for all vectors:

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (Associative law)

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (Commutative law)

$\mathbf{A}\bullet(\mathbf{B}+\mathbf{C}) = \mathbf{A}\bullet\mathbf{B} + \mathbf{A}\bullet\mathbf{C}$ (Distributive Law)

2-5 If $\mathbf{A} = 2\hat{\mathbf{a}}_x - 3\hat{\mathbf{a}}_y + 2\hat{\mathbf{a}}_z$ at all points P,

a) find the expression for **A** in the cylindrical coordinate system.
b) evaluate this expression at the points $P_1(1,60°,2)$ and $P_2(2,30°,4)$.

2-6 If $\mathbf{A} = 2r\,\hat{\mathbf{a}}_r - 3r\sin\phi\,\hat{\mathbf{a}}_\theta$, find the representation of **A** in the Cartesian coordinate system.

2-7 The representation of a vector **C** using the Cartesian coordinate system base vectors is $\mathbf{C} = 3\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y - 3\hat{\mathbf{a}}_z$. Find its representation using the following base vectors:

$$\hat{\mathbf{a}}_1 = \frac{1}{\sqrt{2}}\,[\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_z\,]$$

$$\hat{\mathbf{a}}_2 = \frac{1}{\sqrt{2}}\,[\hat{\mathbf{a}}_x - \hat{\mathbf{a}}_z\,]$$

$$\hat{\mathbf{a}}_3 = \hat{\mathbf{a}}_y \quad .$$

2-8 A force $\mathbf{F} = 10\,\hat{\mathbf{a}}_x - 8\,\hat{\mathbf{a}}_y$ [N] is applied to an object that constrained to travel towards increasing values of x along the path defined by $y = x^2$, $z = 0$. Find the component of $\mathbf{F}$ that is tangent to this path at the point (2,4,0).

2-9 Using integration, calculate the area of the triangular are shown in Figure P2-9.

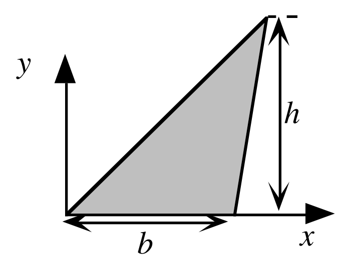

Figure P2-9

2-10 Using integration, find the volume of the right pyramid shown in Figure P2-10.

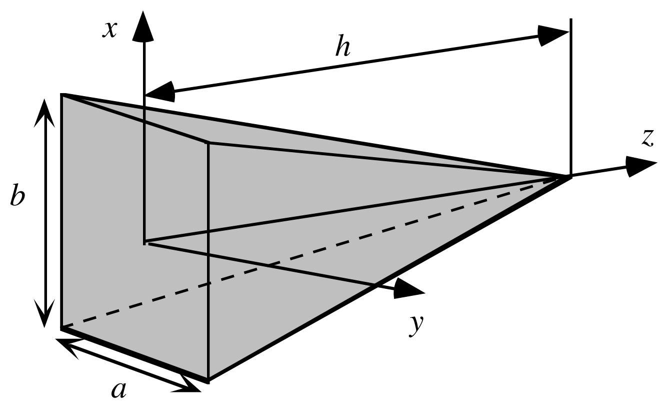

Figure P2-10

2-11 Evaluate the integral $\int_S \mathbf{D}\bullet d\mathbf{s}$ when $\mathbf{D} = r\sin\theta\hat{\mathbf{a}}_r + r\sin\theta\hat{\mathbf{a}}_\theta$ and S is a unit sphere, centered at the origin.

2-12 Consider the integral $\int_C \mathbf{F}\bullet d\ell$, where $\mathbf{F} = \rho\,\hat{\mathbf{a}}_\rho + z^2\,\hat{\mathbf{a}}_\phi$.

a) Calculate this integral from $P(1,0°,0)$ to $P(1,90°,2)$ along the path C_1 shown in Figure P2-12 , which consists of the arc $\rho = 1, 0<\phi<\pi/2, z = 0$, followed by the straight line $\rho = 1, \phi = \pi/2, 0<z < 1$.

b) Calculate this integral from $P(1,0°,0)$ to $P(1,90°,2)$ along the path C_2 shown in Figure P2-12 that is defined by the arc: $\rho = 1$, $0<\phi<\pi/2$, $z = 4\phi/\pi$

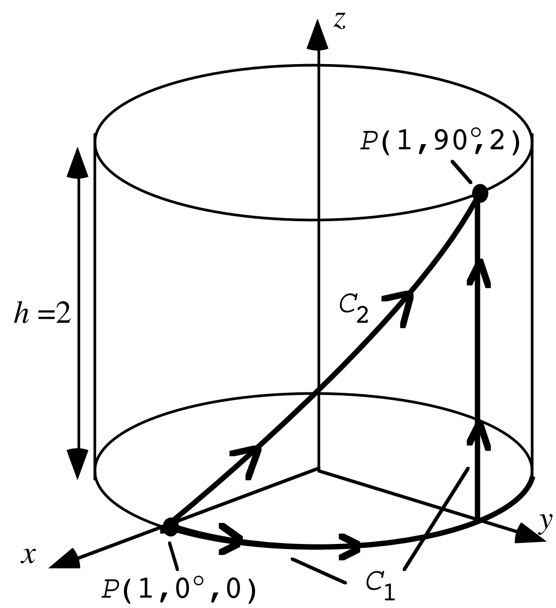

Figure P2-12

2-13 Consider the line integral $\int_C \mathbf{E}\bullet d\ell$, where $\mathbf{E} = x\,\hat{\mathbf{a}}_x + 2xy\,\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z$.

a) Calculate this integral along the path C_1 that extents from the origin to the point $P(1,1,1)$ along the straight line segments that sequentially passes through the points $P(0,0,0)$, $P(1,0,0)$, and $P(1,1,0)$, and $P(1,1,1)$.

b) Calculate this integral along the path C_2 that extents from the origin to the point $P(1,1,1)$ along a single, straight line.

2-14 Evaluate the volume integral $\int_V Q\,dv$, where $Q = 2x^3 z$ when x and z are specified in meters and V is the cube shown Figure P2-14.

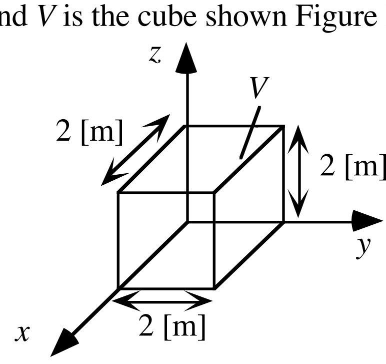

Figure P2-14

2-15 Evaluate the surface integral $\int_S g\,ds$ over the sector shown in Figure P2-15 if $g = 2\rho\cos\phi$.

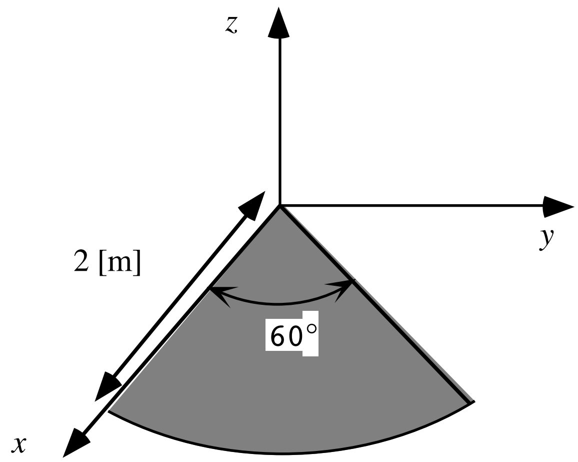

Figure P2-15

2-16 If $\mathbf{F} = xy\,\hat{\mathbf{a}}_x - y\,\hat{\mathbf{a}}_y$, calculate the value of the line integral $\int \mathbf{F}\bullet\mathbf{d}\ell$ from P_1 to P_2 in Figure P2-16 along the paths:

a) along a straight line from P_1 to P_2,
b) along the path P_1AP_2.

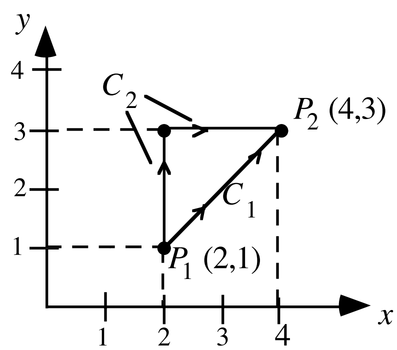

Figure P2-16

2-17 A family of surfaces is defined by the equation

$$2x^2y + xz = C,$$

where each surface corresponds to a different value of the constant C. Find the unit vector $\hat{\mathbf{a}}_n$ that is directed outward from the surface at the point P(1,2,-1).

2-18 For the scalar function $g = 2xy + z^2$, find

a) the magnitude and direction of the maximum rate of change of g at the point P(1,3,2).
b) the rate of change of g along the line directed from P(1,3,2) to P(2,2,-1).

2-19 Consider the line integral $W = \int_{P_1}^{P_2} \mathbf{F} \cdot \mathbf{d}\ell$, where $\mathbf{F} = 4y\,\hat{\mathbf{a}}_y$.

a) Prove that the value of W is independent of the path chosen between the endpoints P_1 and P_2 .

b) Find the value of W when the endpoints are $P_1(1,0,0)$ and $P_2(2,-1,4)$.

2-20 For the function $f = 2xy$,

a) Calculate ∇f in Cartesian coordinates.

b) Express f in cylindrical coordinates and calculate ∇f in cylindrical coordinates

c) Show that ∇f is the same vector in both coordinate systems by transforming the vector found in a) into cylindrical coordinates.

2-21 If the representation of a vector **A** in spherical coordinates is $\mathbf{A} = r\,\hat{\mathbf{a}}_r$,

a) Calculate $\nabla \cdot \mathbf{A}$ in the spherical coordinate system

b) Find the representation of **A** in the Cartesian coordinate system and then calculate $\nabla \cdot \mathbf{A}$. Is it the same value as found in part a)? Why or why not?

2-22 Evaluate the integral $\oint_S \mathbf{D} \cdot \mathbf{ds}$ over the surface bounding the cube shown in Figure P2-14 when $\mathbf{D} = 2y\,\hat{\mathbf{a}}_x + xz\,\hat{\mathbf{a}}_y + z\,\hat{\mathbf{a}}_z$. Show that the same result is obtained using the divergence theorem by integrating $\nabla \cdot \mathbf{D}$ throughout the volume.

2-23 Consider the line integral $\oint_C \mathbf{B} \cdot \mathbf{d}\ell$, where $\mathbf{B} = y\,\hat{\mathbf{a}}_x + z\,\hat{\mathbf{a}}_y$ and C is a square path in the $z = 0$ plane with sides $x = -1$, $x = 1$, $y = -1$ and $y = 1$. Assume that the direction of the path is counterclockwise when looking downward from the $+z$ axis.

a) Calculate the line integral directly.

b) Calculate the line integral by using Stoke's theorem and integrating $\nabla \times \mathbf{B}$ over the square surface in the $z = 0$ plane that is bounded by C.

2-24 Find $\nabla \cdot \mathbf{B}$ and $\nabla \times \mathbf{B}$ if

a) $\mathbf{B} = \rho z\, \hat{\mathbf{a}}_\rho + \rho^2\, \hat{\mathbf{a}}_\phi + 2z^2\, \hat{\mathbf{a}}_z$

b) $\mathbf{B} = 2xy\, \hat{\mathbf{a}}_x + 3y\, \hat{\mathbf{a}}_z$

c) $\mathbf{B} = 4r \sin\theta\, \hat{\mathbf{a}}_r + 3r \cos\phi\, \hat{\mathbf{a}}_\theta$

2-25 Given that $f = r \sin\theta \cos\phi$, calculate

a) ∇f

b) $\nabla \times \nabla f$

c) $\nabla \cdot \nabla f$

2-26 In Figure P2-26, S_1 is a circular disk with unit radius, centered in the $z = 0$ plane, and S_2 is a hemisphere for $z > 0$, centered at the origin with unit radius. If $\mathbf{A} = 3r\, \hat{\mathbf{a}}_\phi$, calculate

$\oint \nabla \times \mathbf{A} \cdot d\mathbf{s}$ S_1 and then on S_2. Assume that the normal direction to both surfaces points has a positive z component. Do these integrals have the same values? Why or why not?

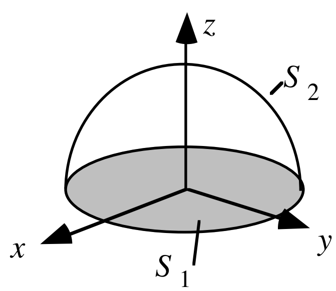

Figure P2-26

2-27 Using the Cartesian coordinate system, verify that the identity $\nabla \times \nabla V = 0$ is valid for all scalar fields V.

2-28 Using the Cartesian coordinate system, verify that the identity $\nabla \cdot \nabla \times \mathbf{A} = 0$ is valid for all vector fields $\mathbf{A}$.

2-29 Using thge identity $\nabla^2 \mathbf{A} = \hat{\mathbf{a}}_x \nabla^2 A_x + \hat{\mathbf{a}}_y \nabla^2 A_y + \hat{\mathbf{a}}_z \nabla^2 A_z$, prove that

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \ .$$