The Inner Product

So we **now** know that a continuous, analog signal $v(t)$ can be expressed as:

$$v(t) = \sum_n a_n \psi_n(t)$$

So that a continuous, **analog** signal can be (almost) completely specified by a **discrete set** of numbers:

$$\{a_1, a_2, a_3, a_4, a_5, a_6, \cdots\}$$

Q: *But don't these numbers likewise* ***depend*** *on the basis functions $\psi_n(t)$?? How is this any easier or* ***simpler*** *than just specifying $v(t)$.*

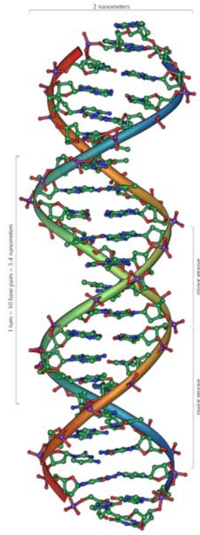
A: Remember, the signal $v(t)$ is **arbitrary**, but the basis functions $\psi_n(t)$ are typically **well-known** and frequently used.

We can think of the basis functions $\psi_n(t)$ as a standard **set of parts**, from which we can construct any **arbitrary** function $v(t)$!

Q: *We think you've gone off the deep end.* ***Parts*** *used for constructing* ***functions****? Isn't this discussion impractical, ephemeral, esoteric and didactic ?*

A: Not at all!

The concept of constructing **massive, complex** things out of small and **simple** elements is **pervasive** not only in engineering, but in other sciences and human activity as well!!



Some examples:

1. The **genetic code** in human DNA is made up of very long sequence of just four purine compounds: adenine (**A**), thymine (**T**), guanine (**G**), and cytosine (**C**).

2. Almost all human knowledge and emotion can be expressed in English using the symbols:

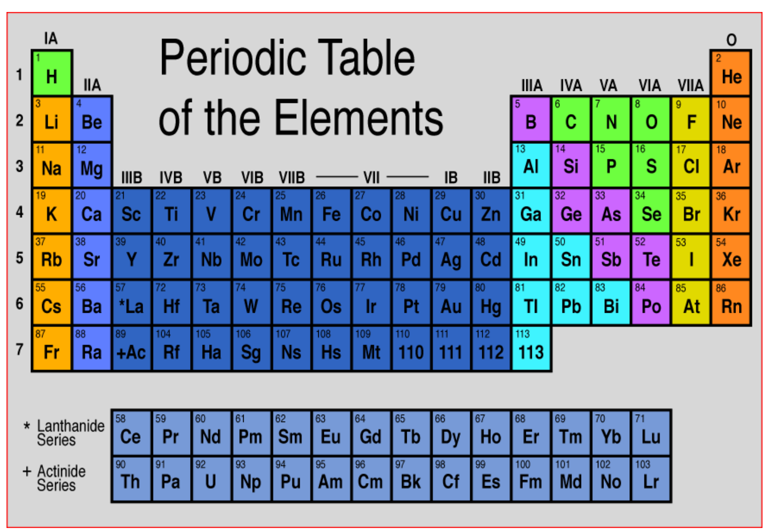
ABCDEFGHIJKLMNOPQRSTU VWXYZ1234567890!,.:;"+-

3. All matter in the universe is constructed with a relative small number of **elements**.

H_2O

$C_6H_{12}O_6$

CO_2



Thus, the set of basis functions $\psi_n(t)$ are the **"parts"** we use to construct our signal $v(t)$. The values $\{a_1, a_2, a_3, \cdots\}$ tell us the specific **"blueprint"** (or recipe, code, paragraph—choose your analogy) for putting these parts together to create the unique function $v(t)$.

Similar to other aspects of life, we must **choose** which set of basis functions are most **useful** to us.

However, instead of choosing between lego blocks and tinker toys, or English, Chinese and Spanish, we must choose between (for example) sinusoids, sinc function, and wavelets!

Q: *But after we choose a basis $\psi_n(t)$, how do we **determine** the **values** a_n? How do we find the "recipe" for constructing function $v(t)$??*

A: First, we must understand what the values a_n tell us about the signal $v(t)$. Essentially, the values a_n tell us **how much** of each basis function $\psi_n(t)$ exists within $v(t)$.

For example, if the value a_1 is small, the value a_2 is moderate, and a_3 is big, or recipe (metaphorically speaking) might be:

*"To create $v(t)$, add a **pinch** of basis function $\psi_1(t)$, a **cup** of basis function $\psi_2(t)$, and about a **gallon** of basis function $\psi_3(t)$. Place in a hot oven for about 45 minutes*."*

* *This last sentence is **not** part of the analogy.*

Q: *I thought I would never say this, but can you be more* ***mathematically specific****?*

A: The values a_n are the **components** of signal $v(t)$, with respect to basis $\psi_n(t)$.

Q: *In EECS 220 we* ***enjoyed*** *learning about the* ***components*** *of a vector (A_x, A_y, A_z), with respect to some set of* ***basis vectors*** *$(\hat{a}_x, \hat{a}_y, \hat{a}_z)$. E.G.,:*

$$\mathbf{A} = A_x\,\hat{\mathbf{a}}_x + A_y\,\hat{\mathbf{a}}_y + A_z\,\hat{\mathbf{a}}_z$$

Is this the ***same*** *thing?*

A: **Precisely** the same thing!

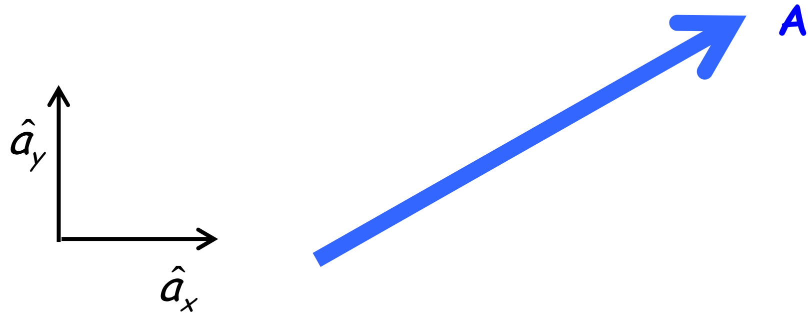
The scalar **components** (A_1, A_2, A_3) of a vector tells us how **much** of each base vector $(\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$ exists within vector **A**.

The scalar components (A_1, A_2, A_3) thus provide the **recipe** for constructing vector **A** from our fundamental "ingredients" $(\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$.

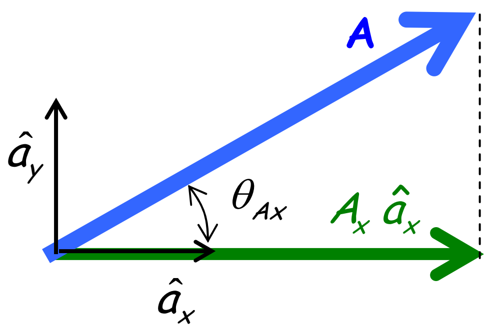
Q: *But wait! I remember that we determined these components (A_1, A_2, A_3) by using the* ***dot product****, e.g.,:*

$$A_x = \mathbf{A}\cdot\hat{\mathbf{a}}_x \qquad A_y = \mathbf{A}\cdot\hat{\mathbf{a}}_y \qquad A_z = \mathbf{A}\cdot\hat{\mathbf{a}}_z$$

A: That's correct! For example, consider a vector **A**, and basis vectors $\hat{\mathbf{a}}_x$ and $\hat{\mathbf{a}}_y$.



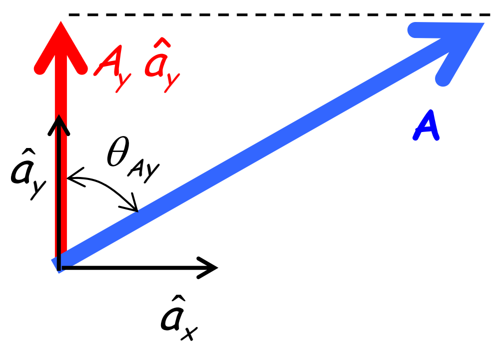
By using the **dot product**, we find the component of vector **A** in the direction of basis vector $\hat{\mathbf{a}}_x$.



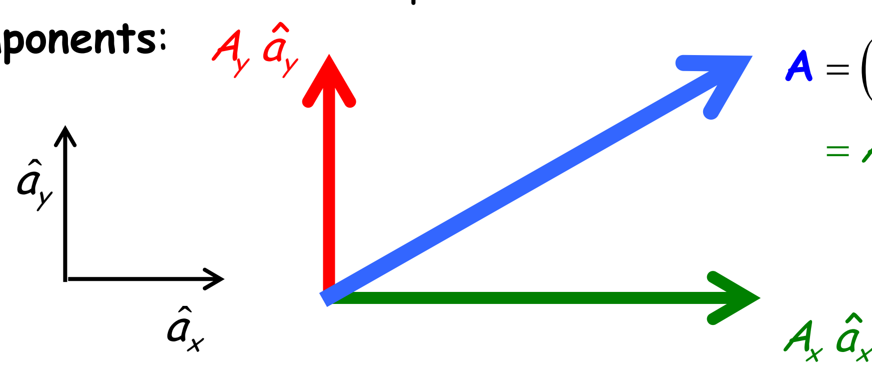
$$
\begin{aligned}
A_x &= |\mathbf{A}| \cos\theta_{Ax} \\
&= |\mathbf{A}| \, |\hat{\mathbf{a}}_x| \cos\theta_{Ax} \\
&= \mathbf{A} \cdot \hat{\mathbf{a}}_x
\end{aligned}
$$

Likewise for the **component** of vector **A** in the direction of basis vector $\hat{\mathbf{a}}_y$.

$$
\begin{aligned}
A_y &= |\mathbf{A}| \cos\theta_{Ay} \\
&= |\mathbf{A}| \, |\hat{\mathbf{a}}_y| \cos\theta_{Ay} \\
&= \mathbf{A} \cdot \hat{\mathbf{a}}_y
\end{aligned}
$$



Thus vector **A** can be expressed as the **sum** of two **vector components**:



$$
\begin{aligned}
\mathbf{A} &= \left(\mathbf{A} \cdot \hat{a}_x\right)\hat{a}_x + \left(\mathbf{A} \cdot \hat{a}_y\right)\hat{a}_y \\
&= A_x \hat{a}_x + A_y \hat{a}_y
\end{aligned}
$$

Q: *But we're **not** talking about vectors, we're talking about **continuous signals** like $v(t)$. **Surely** there's no way to use the dot product to find its components?*

A: Essentially there **is**! The mathematical cousin of the dot product is a mathematical operation known as the **inner product**.*

The **inner product** of two signals $a(t)$ and $b(t)$ is defined as:

$$\langle a(t), b(t) \rangle \equiv \int_{-\infty}^{\infty} a(t) b^*(t) dt$$

where * indicates **complex conjugate**.

If $\langle a(t), b(t) \rangle = 0$, we say that the two signals $a(t)$ and $b(t)$ are **orthogonal** (just like if $\mathbf{A} \cdot \mathbf{B} = 0$!). If this is the case, the two signals $a(t)$ and $b(t)$ are considered to be completely **dissimilar**—they have **no common** component.

The energy of some signal is determined by taking the inner product of that signal with **itself** (just like if $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$!):

$$E_a = \langle a(t), a(t) \rangle \equiv \int_{-\infty}^{\infty} a(t) a^*(t) dt = \int_{-\infty}^{\infty} |a(t)|^2 dt$$

* And stop calling me Shirley.

A signal whose energy is $E = 1$ said to have **unit** energy (just like unit vectors where $\hat{a} \cdot \hat{a} = 1$!).

Finally, consider a set of signals ($\psi_n(t)$, say). These signals each have **unit** energy.

$$\langle \psi_n(t), \psi_n(t) \rangle = 1 \quad \text{for all } n$$

Likewise, these signals are all **orthogonal** to each other:

$$\langle \psi_n(t), \psi_m(t) \rangle = 0 \quad \text{for all } m \neq n$$

These signals are known as **mutually orthogonal**.

A set of mutually orthogonal signals with unit energy is known as an **orthonormal set** of basis functions (just like $\hat{a}_x, \hat{a}_y, \hat{a}_z$).

Orthonormal basis functions have many exceptionally **attractive** mathematical properties. As a result, we find that we frequently use them in **signal expansions** of the form:

$$v(t) = \sum_n a_n \, \psi_n(t)$$

Q: *So does that mean that the **Fourier** and **sinc** basis functions are **orthonormal?***

A: Absolutely! As are many (but not all) **wavelet** basis functions.

The key property of orthonormal basis functions is that it allows us to determine the **signal components** by use of the **inner product**:

$$a_n = \langle v(t), \psi_n(t) \rangle$$

Of course, this is perfectly **analogous** to our vector component analysis:

$$A_n = \mathbf{A} \cdot \hat{\mathbf{a}}_y$$

So now we can determine the values a_n for our most **popular** basis functions!

1. Fourier

$$v(t) = \sum_{n=-\infty}^{\infty} a_n \, \psi_n(t) = \sum_{n=-\infty}^{\infty} a_n \, e^{j\left(\frac{2\pi n}{T}\right)t} \qquad \text{for } 0 \leq t \leq T$$

Therefore:

$$a_n = \langle v(t), \psi_n(t) \rangle = \left\langle v(t), e^{+j\left(\frac{2\pi n}{T}\right)t} \right\rangle = \int_0^T v(t) e^{-j\left(\frac{2\pi n}{T}\right)t} dt = V(\omega_n)$$

2. Sinc Function

$$v(t) = \sum_n a_n \, \psi_n(t) = \sum_n a_n \, sinc\left[\frac{t - n\tau}{\tau}\right]$$

Therefore:

$$a_n = \langle v(t), \psi_n(t) \rangle = \left\langle v(t), sinc\left[\frac{t - n\tau}{\tau}\right] \right\rangle = \int_{-\infty}^{\infty} v(t)\, sinc\left[\frac{t - n\tau}{\tau}\right] dt$$

Q: *Yikes! This integral looks as* ***ugly and unpleasant*** *!*

A: It does **look** that way—but it's **not**! ICBST (It Can Be Shown That) the solution to this integral is **simple** and straight forward:

$$a_n = \int_{-\infty}^{\infty} v(t)\, sinc\left[\frac{t - n\tau}{\tau}\right] dt = v(t = n\tau)$$

The component value a_n is simply the **value** of function $v(t)$ **at** the specific **time** $t = n\tau$. Thus:

$$v(t) = \sum_n a_n \; sinc\left[\frac{t - n\tau}{\tau}\right] = \sum_n v(t = n\tau) \; sinc\left[\frac{t - n\tau}{\tau}\right]$$

This is the reason why sinc basis functions are **so popular**—it is **extremely easy** to determine **all** the signal components a_n!

All we need is a device that **samples** the signal $v(t)$ at specific **times** $t = n\tau$—and we **have** such a device!