Relativistic effects in earth-orbiting Doppler lidar return signals

Neil Ashby\textsuperscript{1,2,*}

\textsuperscript{1}Department of Physics, University of Colorado, Boulder, Colorado 80309-0390, USA
\textsuperscript{2}Present address: Time and Frequency Division, Mail Stop 847, National Institute of Standards and Technology, Boulder, Colorado 80302, USA
\textsuperscript{*Corresponding author: ashby@boulder.nist.gov

Received February 2, 2007; revised June 27, 2007; accepted July 11, 2007; posted August 20, 2007 (Doc. ID 79581); published October 18, 2007

Frequency shifts of side-ranging lidar signals are calculated to high order in the small quantities \( (v/c) \), where \( v \) is the velocity of a spacecraft carrying a lidar laser or of an aerosol particle that scatters the radiation back into a detector \((c\text{ is the speed of light})\). Frequency shift measurements determine horizontal components of ground velocity of the scattering particle, but measured fractional frequency shifts are large because of the large velocities of the spacecraft and of the rotating earth. Subtractions of large terms cause a loss of significant digits and magnify the effect of relativistic corrections in determination of wind velocity. Spacecraft acceleration is also considered. Calculations are performed in an earth-centered inertial frame, and appropriate transformations are applied giving the velocities of scatterers relative to the ground. © 2007 Optical Society of America

OCIS codes: 010.1100, 010.3640, 350.5720.

1. INTRODUCTION

Side-ranging laser imaging detection and ranging (lidar) measurements of wind from space will probably use a laser transmitter in a spacecraft to illuminate aerosol particles or molecules that scatter the radiation almost exactly back toward the transmitter. The measured frequency shift can be processed to yield the velocity of the scattering particles relative to the ground. This paper discusses relativistic corrections to the determination of the wind velocity, arising from large spacecraft velocities, earth rotation, and acceleration of the spacecraft toward the earth. Three different reference frames need to be discussed in this connection: these are the instantaneous rest frame of the lidar apparatus, called the lidar rest frame (LRF), the earth-centered locally inertial frame, called the earth-centered inertial (ECI) frame; and a reference frame at rest relative to the surface of the earth at the point where the wind is measured, called the earth reference frame (ERF).

We assume that the position, velocity, and acceleration of the spacecraft are known in the ECI frame because it seems most likely that the spacecraft would be tracked, and its orbit determined, in such a frame. To simplify the notation, we shall denote quantities measured in the ECI frame without primes. We shall regard the ECI frame, a notation, we shall denote quantities measured in the ECI and its orbit determined, in such a frame. To simplify the seems most likely that the spacecraft would be tracked,

The outline of the paper is as follows. In Section 2 we derive the fractional frequency shift observed in the LRF, assuming the velocity of the apparatus is constant during the propagation time of the radiation. First-order relativistic corrections are discussed. Section 3 develops Lorentz transformations in vector form, needed in Section 4 to derive higher-order relativistic corrections. We use Lorentz transformations in vector form as a compact way of expressing relativistic transformations among the various reference frames \([1–3]\). This is useful when the relative velocities of the reference frames are in arbitrary directions. The vector notation is explained in Section 3. Nu-
Numerical simulations of measurement sequences and measurement processing with relativistic corrections to extract the horizontal component of the wind are discussed in Section 5. Effects due to spacecraft acceleration are derived in Section 6 from the point of view of the ECI frame, and these and similar higher-order relativistic corrections are discussed in Section 7. In Section 8 an alternative derivation of the frequency shift is given neglecting the relative rotation of the ERF and LRF axes. Relativistic corrections of order \(\frac{v}{c}^2\) and \(\frac{v}{c}^3\) are discussed.

Notation. The existence of three or more reference frames in this problem (there is also an instantaneous rest frame for the wind, but such a frame is not used in this paper) results in a notation that is cumbersome in some respects. We denote the velocity of the ground point relative to the ECI (see Figs. 1 and 2) by \(V_E\), and the velocity of the lidar relative to the ECI by \(V_L\). We assume that in a single measurement the radiation is incident on the aerosol particle for such a short time that \(w/c\) is constant. However, during the few milliseconds (ms) of propagation time from the lidar to the scatterer and back to the detector, the velocity of the detector relative to the ECI may have changed. This is fully discussed in Section 6.

In the calculation given in the following section, we regard the lidar apparatus as being at rest. The aerosol particle is assumed to have velocity \(w'\) relative to the lidar apparatus.

### Table 1. Description of Frames of Reference

<table>
<thead>
<tr>
<th>Reference Frame</th>
<th>Origin</th>
<th>Axis Directions</th>
<th>Velocity in ECI</th>
<th>Rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>ECI</td>
<td>Earth’s center</td>
<td>Fixed in space</td>
<td>0</td>
<td>None</td>
</tr>
<tr>
<td>LRF</td>
<td>Lidar transmitter</td>
<td>Fixed in space</td>
<td>(V_L)</td>
<td>None</td>
</tr>
<tr>
<td>ERF</td>
<td>Ground point</td>
<td>Fixed in space</td>
<td>(V_E)</td>
<td>None</td>
</tr>
<tr>
<td>ECEF</td>
<td>Earth’s center</td>
<td>Fixed in earth</td>
<td>0</td>
<td>Rotating with earth</td>
</tr>
</tbody>
</table>

In Figure 4, the scattering particle is represented as an irregular object that moves with velocity \(w'\) in the \(x'\) direction. The wave vector \(k'\) of the incident wave has \(x'\) component \(k' \cos \theta'\), so that \(w'k' \cos \theta' = w' \cdot k''\). Two wavefronts for the incident wave are shown, the first labeled #1 incident that strikes the particle at some initial time \(t''\), and another one labeled #2 incident, a distance \(\lambda_0' = 2\pi/k''\) away, that will hit the moving scatterer a time \(\Delta t''\) later. Figure 5 shows first scattered wavefront #1 reflected as though it were plane, at time \(\Delta t''\) later. During the time \(\Delta t''\) the scatterer moves a distance \(w'' \Delta t''\) in the positive \(x''\) direction. At this instant the second wavefront impinges on the particle; this event would generate a second backscattered wavefront. Meanwhile the first back-
scattered wave propagates exactly back toward the transmitter a distance \( c\Delta t'' \). The two backscattered wavefronts are separated by the wavelength \( \lambda'' \).

The component of the distance moved by the scattering particle in the direction of the incident ray during time \( \Delta t'' \) is \( w'' \cos \theta'' \Delta t'' \). The distance between the two incident waves is the wavelength \( \lambda'_0 \) of the incident wave, and from Figure 5 it is seen that

\[ \lambda'_0 = c \Delta t'' - w'' \cos \theta'' \Delta t''. \]  

(1)

For the ray scattered back toward the detector, the wavelength is the distance between reflected wave \#1 and incident wave \#2. Denoting the wavelength of the scattered photons by \( \lambda'' \), we have

\[ \lambda'' = c \Delta t'' + w'' \cos \theta'' \Delta t''. \]  

(2)

Combining the two equations for the incident and scattered wavelength,

\[ \frac{\lambda'_0}{\lambda''} = \frac{c(\Delta t'')}{c(\Delta t'')} = \frac{c \Delta t'' - w'' \Delta t'' \cos \theta''}{c \Delta t'' + w'' \Delta t'' \cos \theta'}. \]  

(3)

Therefore the ratio of the backscattered frequency to the incident frequency is

\[ \frac{f''}{f'_0} = \frac{1}{c} \left( \frac{w''}{c} \right) \cos \theta'' \cdot \frac{1}{c} \left( \frac{w''}{c} \right) \cos \theta'' \]  

\[ = \frac{1}{c} \left( \frac{w''}{c} \right) \left( \frac{w''}{c} \right) \cos \theta'' \cos \theta''. \]  

(4)

This result applies when the lidar velocity is constant; a special case has been discussed by Gudimetla and Kavaya [4]. Introducing the measured fractional frequency shift \( \Delta f''/f'_0 \) by means of

\[ \frac{f''}{f'_0} = 1 + \frac{\Delta f''}{f'_0}, \]  

(5)

we may solve for the quantity involving the wind velocity \( w'' \) to obtain

\[ \frac{w'' \cdot k''}{ck''} = \frac{1}{2 f'_0} \left( \frac{\Delta f''}{f'_0} \right). \]  

(6)

Thus on the right-hand side of the above equation, only the measured fractional frequency shift is needed in order to determine the wind velocity. If the measurements can be processed in such a way that only the fractional frequency shift is involved, then no error in the wind velocity will occur resulting from a lack of knowledge of the actual transmitter frequency.

If we denote the unit vector pointing from transmitter to scatterer by \( \mathbf{n}' = k''/k'' \), this result can be written in the form

\[ \mathbf{w}'' \cdot \mathbf{n}' = \frac{1}{2 f'_0} \left( \frac{\Delta f''}{f'_0} \right). \]  

(7)

The term in the denominator is a relativistic correction since it is \( c^{-1} \) times the numerator. Equation (7) could be expanded to second order in the fractional frequency shift, giving

\[ \mathbf{w}'' \cdot \mathbf{n}' = -\frac{c}{2 f'_0} \left( \frac{\Delta f''}{f'_0} \right) + \frac{c (\Delta f''/f'_0)^2}{2 f'_0} + \cdots \]  

(8)

For a spacecraft velocity of 7 km/s, the correction represented by the second term in Eq. (8) can contribute more than 0.15 m/s to the wind velocity.

The right-hand side of Eq. (7) is determined by the measurement itself. Thus the component of the wind velocity in the direction of the beam is determined in the LRF. The goal, however, is to obtain the velocity relative to the ground, that is, to find \( \mathbf{w} \) in the ERF. Simple Galilean transformations lead to appreciable error; instead we need the Lorentz velocity transformations.

Equation (7) gives the measured fractional frequency shift in the LRF, and an observable quantity having the fractional frequency shift as a relativistic correction in the denominator has been derived. Since the quantity of interest is the wind or aerosol velocity in the earth-fixed
frame, the results must be transformed into the ERF. Additional relativistic effects enter when these transformations are applied.

3. LORENTZ TRANSFORMATIONS IN VECTOR FORM

We now prepare to derive a general expression for the wind velocity, not restricted to leading-order relativistic corrections. In this section we introduce the vector form of the Lorentz transformations, which is a compact way of writing the transformations. This form of the Lorentz transformations can be found in many textbooks [1–3].

A general Lorentz transformation of position can be separated into a part parallel to the relative velocity, which suffers Lorentz contraction, and a part perpendicular to the relative velocity, which is unchanged. Since coordinate differences are of interest in the derivations that follow, constant translations of origins are not significant. We shall discuss the transformation from ECI to ERF coordinates with parallel axes. Let \( \mathbf{r} \) be a position vector in the ECI frame, and denote position vectors and time measurements in the ERF with primes. Then the component of \( \mathbf{r}' \) parallel to \( \mathbf{V}_E \) is

\[
\frac{\mathbf{V}_E\mathbf{r}' \cdot \mathbf{V}_E}{V_E^2},
\]

and this part of the vector suffers Lorentz contraction, expressed in the transformation equations by multiplying by the factor

\[
\gamma_E = \frac{1}{\sqrt{1-V_E^2/c^2}}. \tag{10}
\]

The component of \( \mathbf{r}' \) perpendicular to the velocity \( \mathbf{V}_E \) is

\[
\mathbf{r}' - \frac{\mathbf{V}_E\mathbf{r}' \cdot \mathbf{V}_E}{V_E^2}. \tag{11}
\]

Then the Lorentz transformations in vector form are

\[
\mathbf{r} = \mathbf{r}' - \frac{\mathbf{V}_E\mathbf{r}' \cdot \mathbf{V}_E}{V_E^2} + \gamma_E \left( \frac{\mathbf{V}_E\mathbf{r}' \cdot \mathbf{V}_E}{V_E^2} + \frac{\mathbf{V}_E}{c}ct' \right), \tag{12}
\]

\[
ct' = \gamma_E \left( ct - \frac{\mathbf{V}_E \cdot \mathbf{r}}{c} \right), \tag{13}
\]

since only the component of \( \mathbf{r}' \) parallel to \( \mathbf{V}_E \) suffers Lorentz contraction, and only the component of \( \mathbf{r}' \) parallel to the velocity enters the time transformation. Dotting \( \mathbf{V}_E \) into Eq. (12),

\[
\mathbf{V}_E \cdot \mathbf{r} = \gamma_E \left( \mathbf{V}_E \cdot \mathbf{r}' + \frac{V_E^2}{c}ct' \right). \tag{14}
\]

The inverses of these transformation equations are

\[
r' = \mathbf{r} + (\gamma_E - 1) \frac{\mathbf{V}_E\mathbf{r}}{V_E^2} - \gamma_E \frac{\mathbf{V}_E}{c}ct',
\]

and

\[
V_E \cdot \mathbf{r}' = \gamma_E \left( V_E \cdot \mathbf{r} - \frac{V_E^2}{c}ct \right). \tag{15}
\]

We shall also need velocity transformations, obtained by taking differentials of both sides of the position and time transformations. Thus, for example, from Eqs. (15),

\[
d\mathbf{r}' = d\mathbf{r} + (\gamma_E - 1) \frac{\mathbf{V}_E\mathbf{V}_E \cdot d\mathbf{r}}{V_E^2} - \gamma_E \frac{\mathbf{V}_E}{c}c\, dt; \tag{16}
\]

\[
c\, dt' = \gamma_E \left( c\, dt - \frac{\mathbf{V}_E \cdot d\mathbf{r}}{c} \right), \tag{17}
\]

and taking the ratio of corresponding sides of the above two equations, we obtain the vector form of the velocity transformation equations for \( \mathbf{v}' \) in terms of \( \mathbf{v} \) and \( \mathbf{V}_E \):

\[
\frac{d\mathbf{r}'}{d\mathbf{t}'} = \frac{\mathbf{d} \mathbf{r} + (\gamma_E - 1) \frac{\mathbf{V}_E\mathbf{V}_E \cdot d\mathbf{r}}{V_E^2} - \gamma_E \frac{\mathbf{V}_E}{c}c\, dt}{\gamma_E \left( d\mathbf{t} - \frac{\mathbf{V}_E \cdot d\mathbf{r}}{c} \right)}
\]

\[
= \frac{\sqrt{1-V_E^2/c^2} \mathbf{V}_E + (1 - \sqrt{1-V_E^2/c^2}) \frac{\mathbf{V}_E\mathbf{V}_E \cdot \mathbf{v}}{V_E^2} - \mathbf{V}_E}{\frac{1}{1 - \sqrt{1-V_E^2/c^2}}}, \tag{18}
\]

where the velocity of an object measured in the ECI frame is \( \mathbf{v} = d\mathbf{r}/d\mathbf{t} \). We also need the inverse of Eq. (18):

\[
\frac{\sqrt{1-V_E^2/c^2} \mathbf{v}' + (1 - \sqrt{1-V_E^2/c^2}) \frac{\mathbf{V}_E\mathbf{V}_E \cdot \mathbf{v}'}{V_E^2} + \mathbf{V}_E}{\frac{1 + \mathbf{V}_E \cdot \mathbf{v}'}{c^2}}, \tag{19}
\]

giving the velocity \( \mathbf{v} \) in the ECI frame in terms of the velocity \( \mathbf{v}' \) in the ERF.

4. CONSTANT SPACECRAFT VELOCITY

In this section we give a relativistic treatment of the lidar measurement, assuming that the spacecraft velocity does not change during the time interval between transmission and detection of the signal. We assume for simplicity that the scattering event occurs at a fixed position \( \mathbf{r}'_s \) and time \( t'_s \) in the LRF; that is, that no significant time delay occurs during the interaction between pulse and scatterer.

Also, we assume that the detector is effectively at the same point as the lidar transmitter in the LRF, so that the path of the reflected signal lies exactly on top of the transmitted signal path. Then if \( \mathbf{d}'_s \) is the displacement
from transmitter to scatterer and \( \mathbf{d}^* \) is the displacement from scattering particle to detector, we have
\[
\mathbf{d}^* + \mathbf{d}^* = 0.
\] (20)

The task is to transform Eq. (7) into the ERF, to obtain a useful expression for the wind velocity \( \mathbf{w} \). The process involves transforming first from LRF to ECI and then from ECI to ERF. When two parallel-axis Lorentz transformations in different directions are compounded, the axes of the final frame are no longer necessarily parallel to the axes of the initial frame. This issue is discussed in Section 9.

Suppose that we specify the line of sight (LOS) in the ECI. This is an important case because the spacecraft orbit is most likely to be specified in the ECI, and the ground point position and velocity will be known or can be calculated if the ECI time of the lidar shot is known.

Our starting point is therefore Eq. (7). The vector \( \mathbf{n} \) from the lidar transmitter to the aerosol particle is presumed known in the ECI system. To obtain an expression for \( \mathbf{n}^* \) in terms of \( \mathbf{n} \), use Eq. (15) with \( \mathbf{V}_L \) in place of \( \mathbf{V}_E \) and \( \mathbf{r} \) in place of \( \mathbf{r}' \). Then, for the vector displacement between transmitting and scattering events, we have
\[
\begin{align*}
\mathbf{d}^* &= \mathbf{r}^* - \mathbf{r}^* = \mathbf{r} - \mathbf{r} + (\gamma_L - 1) \frac{\mathbf{V}_L \cdot (\mathbf{r} - \mathbf{r})}{V^2_L} - \frac{\gamma_L}{c} (ct_s - ct_t),
\end{align*}
\] (21)

where
\[
\gamma_L = \frac{1}{\sqrt{1 - \frac{V^2_L}{c^2}}}.
\] (22)

We define the vector from transmitter to scatter by
\[
\mathbf{r} - \mathbf{r} = \mathbf{d}_t.
\] (23)

Then by the constancy of the speed of light, \((ct_s - ct_t) = d_t\), and we have
\[
\begin{align*}
\mathbf{d}^* &= \mathbf{d}_t + (\gamma_L - 1) \frac{\mathbf{V}_L \cdot \mathbf{d}_t}{V^2_L} - \frac{\gamma_L}{c} d_t,
\end{align*}
\] (24)

The magnitude \( d^*_t \) of the displacement vector is obtained by taking the dot product of \( \mathbf{d}^*_t \) with itself. Since there are three terms in Eq. (24), there are basically six terms in the square:
\[
\begin{align*}
(d^*_t)^2 &= \mathbf{d}_t^* \cdot \mathbf{d}_t^* = d_t^2 + 2(\gamma_L - 1) \frac{(\mathbf{V}_L \cdot \mathbf{d}_t)^2}{V^2_L} - 2 \gamma_L \frac{\mathbf{V}_L \cdot \mathbf{d}_t}{V^2_L} d_t + (\gamma_L^2 - 1) \frac{V^2_L (\mathbf{d}_t)^2}{V^2_L} - 2 \gamma_L (\gamma_L - 1) \frac{V^2_L (\mathbf{d}_t)^2}{c} - 2(\gamma_L^2 - \gamma_L) \frac{V^2_L (\mathbf{d}_t)^2}{c} d_t + \frac{\gamma_L^2}{c^2} d_t^2.
\end{align*}
\] (25)

There is considerable cancellation. Then using the identities
\[
\begin{align*}
\frac{\gamma_L^2 - 1}{V^2_L} &= \frac{\gamma_L^2}{c^2}, \quad 1 + \frac{\gamma_L^2 V^2_L}{c^2} = \gamma_L^2,
\end{align*}
\] (26)

the expression becomes a perfect square, giving the result
\[
\mathbf{d}_t^* = \gamma_L d_t \left( 1 - \frac{\mathbf{V}_L \cdot \mathbf{d}_t}{cd_t} \right).
\] (27)

Then we may form an expression for the unit vector \( \mathbf{n}_t^* \) in the LRF in terms of quantities observed or measured in the ECI:
\[
\begin{align*}
\mathbf{n}_t^* &= \mathbf{d}_t^* + (\gamma_L - 1) \frac{\mathbf{V}_L (\mathbf{V}_L \cdot \mathbf{n}_t)}{V^2_L} - \frac{\gamma_L}{c} \mathbf{V}_L \\
&= \frac{\gamma_L}{c} \left( 1 - \frac{\mathbf{V}_L \cdot \mathbf{n}_t}{c} \right)
\end{align*}
\] (28)

where \( \mathbf{n}_t = \mathbf{d}_t / d_t \). Also, it then follows that
\[
\mathbf{V}_L \cdot \mathbf{n}_t^* = \frac{\mathbf{V}_L \cdot \mathbf{n}_t - \frac{V^2_L}{c}}{1 - \frac{\gamma_L}{c}}.
\] (29)

In the following derivation, the left-hand side of Eq. (28) will be used as an abbreviation for the right-hand side, in which the quantities are expressed in the ECI frame.

We want to express \( \mathbf{w}^* \) in terms of \( \mathbf{w}' \) in the ERF. We shall do this in two stages, first transforming with a parallel axis transformation to the ECI frame, and then by a parallel axis transformation to the ERF. The velocity transformation to the ECI is
\[
\mathbf{w} + (\gamma_L - 1) \frac{\mathbf{V}_L \mathbf{V}_L \cdot \mathbf{w}}{V^2_L} - \gamma_L \mathbf{V}_L = \gamma_L \left( 1 - \frac{\mathbf{V}_L \cdot \mathbf{w}}{c^2} \right)
\] (30)

Therefore, dotting \( \mathbf{n}_t^* \) into this expression, we have from Eq. (7)
\[
\begin{align*}
\mathbf{w} \cdot \mathbf{n}_t^* + (\gamma_L - 1) \frac{\mathbf{V}_L \cdot \mathbf{n}_t^* \mathbf{V}_L \cdot \mathbf{w}}{V^2_L} - \gamma_L \mathbf{V}_L \cdot \mathbf{n}_t^* &= \gamma_L \frac{c}{2 f_0} \\
&= \frac{1}{2 f_0} \left( 1 + \frac{\Delta v}{c} \right)
\end{align*}
\] (31)

We manipulate this equation so that all terms involving \( \mathbf{w} \) are isolated as linear terms on the left side. As a first step,
\[
\begin{align*}
\mathbf{w} & \cdot \left[ \mathbf{n}_E + (\gamma_L - 1) \frac{\mathbf{V}_L \cdot \mathbf{n}_E}{V_L^2} \frac{c \Delta f''}{\gamma_L} \right] \\
& = \mathbf{V}_L \cdot \mathbf{n}_E + \frac{c \Delta f''}{1 - \frac{\mathbf{V}_L \cdot \mathbf{w}}{c^2}} \frac{1}{1 + \frac{1}{2} f_0^*}. 
\end{align*}
\]

and then moving the term on the right that has \( \mathbf{w} \) in it to the left,

\[
\begin{align*}
\mathbf{w} & \cdot \left[ \mathbf{n}_E + (\gamma_L - 1) \frac{\mathbf{V}_L \cdot \mathbf{n}_E}{V_L^2} \frac{c \Delta f''}{\gamma_L} \right] \\
& = \mathbf{V}_L \cdot \mathbf{n}_E + \frac{c \Delta f''}{1 - \frac{\mathbf{V}_L \cdot \mathbf{w}}{c^2}} \frac{1}{1 + \frac{1}{2} f_0^*}. 
\end{align*}
\]

To save writing, we introduce the following abbreviation:

\[
\mathbf{A} = \mathbf{n}_E/\gamma_L + (1 - 1/\gamma_L) \frac{\mathbf{V}_L \cdot \mathbf{n}_E}{V_L^2} \frac{c \Delta f''}{\gamma_L} + \mathbf{V}_L - \frac{c \Delta f''}{1 - \frac{\mathbf{V}_L \cdot \mathbf{w}}{c^2}} \frac{1}{1 + \frac{1}{2} f_0^*}. \tag{34}
\]

Then we obtain

\[
\mathbf{w} \cdot \mathbf{A} = \mathbf{V}_L \cdot \mathbf{n}_E + \frac{c \Delta f''}{1 - \frac{\mathbf{V}_L \cdot \mathbf{w}}{c^2}} \frac{1}{1 + \frac{1}{2} f_0^*}. \tag{35}
\]

On the right side of Eq. (35), the quantities are presumed known from the lidar measurement or from computations performed in the ECI frame. Also, on the left, the vector \( \mathbf{A} \) may be presumed known. \( \mathbf{A} \) is almost but not quite a unit vector. Also, although \( \mathbf{A} \) appears to depend on doubly primed quantities, the appearance of \( \mathbf{n}_E \) is to be regarded as shorthand for the substitution represented in Eq. (28).

Inserting Eq. (28) into the definition of \( \mathbf{A} \), after some simplification we obtain the following expression for \( \mathbf{A} \):

\[
\mathbf{n}_E = \frac{\mathbf{V}_L \cdot \mathbf{n}_E}{\gamma_L} = \mathbf{V}_L - \frac{c \Delta f''}{1 - \frac{\mathbf{V}_L \cdot \mathbf{w}}{c^2}} \frac{1}{1 + \frac{1}{2} f_0^*}. \tag{36}
\]

The next step is to extract that part of \( \mathbf{w} \) that depends on \( \mathbf{w}' \), the desired wind velocity in the ERF. The velocity transformation from Eq. (19) is

\[
\mathbf{w} = \frac{\mathbf{V}_E \cdot \mathbf{w'}}{\gamma_E} + \gamma_E \mathbf{V}_E \tag{37}
\]

Dotting \( \mathbf{A} \) into this equation, and using Eq. (35), we have

\[
\mathbf{w} \cdot \mathbf{A} + (\gamma_E - 1) \frac{\mathbf{V}_E \cdot \mathbf{A}}{V_E^2} + \gamma_E \mathbf{V}_E \cdot \mathbf{A} \frac{c \Delta f''}{\gamma_E} \frac{1}{1 + \frac{1}{2} f_0^*} \]

\[
= \mathbf{V}_L \cdot \mathbf{n}_E^* + \frac{c \Delta f''}{1 + \frac{1}{2} f_0^*}. \tag{38}
\]

Again, a term in \( \mathbf{w}' \) occurs in the denominator. Multiplying by the factor in the denominator that has \( \mathbf{w}' \) in it to

\[
\begin{align*}
\mathbf{w}' & \cdot \mathbf{A} = \mathbf{V}_E \left[ (\gamma_E - 1) \frac{\mathbf{V}_E \cdot \mathbf{A}}{V_E^2} + \gamma_E \mathbf{V}_E \cdot \mathbf{A} \frac{c \Delta f''}{\gamma_E} \frac{1}{1 + \frac{1}{2} f_0^*} \right] \\
& = \mathbf{V}_E \left[ (\gamma_E - 1) \frac{\mathbf{V}_E \cdot \mathbf{A}}{V_E^2} \right] \frac{c \Delta f''}{\gamma_E} \frac{1}{1 + \frac{1}{2} f_0^*} + \mathbf{V}_L \cdot \mathbf{n}_E^* - \mathbf{V}_E \cdot \mathbf{A}. \tag{39}
\end{align*}
\]

We introduce an abbreviation for the quantity in square brackets:

\[
\begin{align*}
\mathbf{B} & = \left[ (\gamma_E - 1) \frac{\mathbf{V}_E \cdot \mathbf{A}}{V_E^2} \right] \frac{c \Delta f''}{\gamma_E} \frac{1}{1 + \frac{1}{2} f_0^*} + \mathbf{V}_L \cdot \mathbf{n}_E^* \frac{c \Delta f''}{\gamma_E} \frac{1}{1 + \frac{1}{2} f_0^*}. \tag{40}
\end{align*}
\]

Then the result is

\[
\mathbf{w}' \cdot \mathbf{B} = \frac{c \Delta f''}{2 f_0^*} + \mathbf{V}_L \cdot \mathbf{n}_E^* - \mathbf{V}_E \cdot \mathbf{A}. \tag{41}
\]

The main result, Eq. (41), or appropriate approximations to it, should be used in analyzing Doppler lidar data. We neglected spacecraft acceleration in deriving Eq. (41). Obviously numerous relativistic corrections enter the result. As a first check, let us pass to the nonrelativistic limit, in which all terms involving \( 1/c \) are neglected. Then
\[ \mathbf{n}^\prime = \mathbf{n}, \quad \mathbf{A} = \mathbf{n}, \quad \mathbf{B} = \mathbf{A} = \mathbf{n}. \]

We also neglect the term \( \Delta f''/f'' \) in the denominator and use \( \lambda'' = c/f_0'' \). We obtain

\[ \mathbf{w}' \cdot \mathbf{n}_f = -\frac{\lambda''}{2} + (\mathbf{V}_L - \mathbf{V}_E) \cdot \mathbf{n}_f. \quad (42) \]

Thus the theoretical expressions tell us how to remove the velocities \( \mathbf{V}_L \) and \( \mathbf{V}_E \) from the measurement so that the component of wind velocity \( \mathbf{w}' \) along the LOS is determined. [In Eq. (42) the LOS is specified in the ECI frame.] Since the spacecraft velocity has by far the largest magnitude of all velocities in the problem, such subtractions will typically involve the loss of two or three digits of accuracy. For example, if the wind velocity \( \mathbf{w}' \) were zero, there could still be a large measured Doppler shift due to the large velocity of the spacecraft relative to the ground. Then Eq. (42) shows that most of the observed Doppler shift will be canceled by the term \( \mathbf{V}_L \cdot \mathbf{n}_f \), and the remaining shift will be canceled by the term \( -\mathbf{V}_E \cdot \mathbf{n}_f \).

Equation (42) could be tested by observing the backscattered ground return signal.

Keeping terms of \( O(V_L/c) \) only, we may expand Eq. (36):

\[ \mathbf{A} = \mathbf{n}_f \left( 1 + \frac{\mathbf{V}_L \cdot \mathbf{n}_f}{c} \right) - \frac{\mathbf{V}_L}{c}, \quad (43) \]

\[ \mathbf{B} = \mathbf{A}. \quad (44) \]

The result of substituting these approximations into Eq. (41) is

\[ \mathbf{w}' \cdot \left( \mathbf{n}_f - \frac{\mathbf{V}_L - \mathbf{n}_f \cdot \mathbf{V}_L}{c} \right) = -\frac{c}{2f_0''} \Delta f'' + \left( \mathbf{V}_L - \mathbf{V}_E \right) \cdot \mathbf{n}_f \]

\[ = \frac{\mathbf{V}_L - \mathbf{n}_f \cdot \mathbf{V}_L}{c}. \quad (45) \]

The additional terms added to \( \mathbf{n}_f \) inside the parentheses are an effect of relativistic aberration— the change of apparent direction of a light ray when a transformation between relatively moving reference frames is introduced. Here, the terms \( \mathbf{V}_L - \mathbf{n}_f \cdot \mathbf{V}_L \) are perpendicular to \( \mathbf{n}_f \). Thus the first-order relativistic corrections serve to add terms that slightly change the direction of the LOS. There are several relativistic corrections in Eq. (45); most of them are of comparable orders of magnitude. On the right side of this equation, there are corrections of order \( (V_L/c)^2 V_L \) to the main term, which is \( \mathbf{V}_L \cdot \mathbf{n}_f \). A typical correction contributes about 0.2 ms\(^{-1}\) to the right side, but the net effect depends on the geometry.

Expanding to second order in powers of \( (V_L/c) \) and \( (V_E/c) \) gives

\[ \mathbf{w}' \cdot \left( \mathbf{n}_f - \frac{\mathbf{V}_L - \mathbf{n}_f \cdot \mathbf{V}_L}{c} \right) = -\frac{c}{2f_0''} \Delta f'' + \left( \mathbf{V}_L - \mathbf{V}_E \right) \cdot \mathbf{n}_f \]

\[ + \frac{\mathbf{V}_E \cdot \mathbf{n}_f}{c^2} \left( \frac{(\mathbf{V}_L \cdot \mathbf{n}_f)^2 - V_E^2}{2} \right) \quad (46) \]

The largest second-order terms contribute corrections to the wind speed of order \( (V_L/c)^2 V_L \approx 4 \times 10^{-6} \) m/s, or 5 \times 10\(^{-10}\) rad to the direction of the line of sight. Such small corrections are probably negligible for near-earth applications. In any case, expansion to second or higher order gives expressions that are so cumbersome that it is probably better to use exact expressions in numerical calculations.

Figure 6 shows an example in which there exists a uniform horizontal wind field and a horizontal relative velocity \( \mathbf{V}_L - \mathbf{V}_E \) of the spacecraft relative to the ground. The scanning angle of the lidar is at 30° relative to the nadir.
in the denominator in Eq. (45) results in systematic errors in determining the wind speed that are plotted in Fig. 7 for various scanning angles. In this example \(V_E=300\) ms\(^{-1}\), and the altitude of the lidar was 400 km.

To obtain a better idea of the importance of relativistic corrections, we have developed a numerical simulation of the measurement process involving an earth-orbiting lidar apparatus. This will be explained in the next section and some examples will be discussed.

5. SIMULATIONS: EXAMPLES

In this section we describe numerical simulations using Eq. (41). These examples illustrate how the theory is to be applied and also indicate how large the relativistic corrections can be. The scattering particle is assumed to be on earth’s surface, at the chosen observation point of latitude/longitude \((40.00^\circ N, -105^\circ W)\). At this point the wind is assumed to be constant, \(w^* = (50.00\) m/s E, 47.00 m/s N). We assume the spacecraft is in a Keplerian orbit characterized by the following orbital parameters, from which the position and velocity in the ECI frame can be derived when the time is specified:

\[
R_e = 6.3780000 \times 10^6 \text{ m, earth radius,} \tag{47}
\]

\[
h = 4.0000000 \times 10^5 \text{ m, nominal altitude,} \tag{48}
\]

\[
a = R_e + h = 6.7780000 \times 10^6 \text{ m, orbit semimajor axis,} \tag{49}
\]

\[
e = 0.00200 \text{ eccentricity,} \tag{50}
\]

\[
\omega = 1.600 \text{ rad, altitude of perigee,} \tag{51}
\]

\[
\Omega = -1.309 \text{ rad, angle of ascending line of nodes,} \tag{52}
\]

\[
i = 60.00^\circ, \text{ orbit inclination,} \tag{53}
\]

\[
t_p = 3400.00 \text{ s, time of perigee passage.} \tag{54}
\]

A. First Lidar Measurement

Given the time of perigee passage above, and assuming the Greenwich meridian of earth rotates in the ECI reference frame with angular velocity \(\omega_e = 7.290000 \times 10^{-5}\) rad/s, the azimuthal angle of the ground point as a function of time \(t\) (in the ECI reference frame) is

\[
\phi(t) = -105^\circ + \pi/180 + \omega_e t. \tag{55}
\]

The factors \(\pi/180\) in the first term of the above equation just convert the angle to radians. In a similar manner, any point with given latitude and longitude on earth’s surface can be found in the ECI reference frame.

Also, at any given time, with the Keplerian orbital parameters listed above, the ECI position of the lidar transmitter can be found by solving the Keplerian equations of motion [5]. First one solves for the eccentric anomaly \(E\) in the equation

\[
E - e \sin E = \sqrt{\frac{GM}{a^3}}(t - t_p), \tag{56}
\]

where \(GM=3.986004415 \times 10^{14} \text{ m}^3/\text{s}^2\) is the product of the Newtonian gravitational constant and the earth’s mass. Then the true anomaly \(f\) is found from

\[
\cos f = \frac{\cos E - e}{1 - e \cos E}. \tag{57}
\]

The radius of the spacecraft is obtained from

\[
r = a(1 - e \cos E). \tag{58}
\]

The ECI Cartesian coordinates of the spacecraft are then found from elementary trigonometry:

\[
x = r(\cos \Omega \cos f + \cos \Omega \cos f), \tag{59}
\]

\[
y = r(\sin \Omega \cos f + \sin \Omega \cos f), \tag{59}
\]

\[
z = r \sin I \cos f. \tag{59}
\]

Assume that a pulse from the lidar apparatus starts to the ground point at \(t = t_s = 13750.000\) s exactly. This time is specified in the ECI frame. The position and velocity of the lidar at this moment is obtained by solving the Keplerian equations of motion, and is found to be

\[
r_E = (3490685.517, -4268379.510, 3926595.155) \text{ m,} \tag{60}
\]

\[
V_E = (1415.72415, 5709.87278, 4928.19131) \text{ m/s.} \tag{61}
\]

To find the time \(t_s\) when the pulse arrives at the ground point, one uses constancy of the speed of light:

\[
l_E = \frac{1}{c}|r_s(t_s) - r_l(t_s)|, \tag{62}
\]

where \(r_s(t_s)\) and \(r_l(t_s)\) are the positions of scatterer and transmitter at the scattering and transmission events, respectively. An estimate of the time of arrival \(t_s\) is then used to recompute the ground point position in the ECI frame; then the distance between the ground point and the transmission event is recalculated, giving a better estimate of the arrival time. This process converges after three or four steps and gives for the time of arrival

\[
t_s = 13750.0023732965 \text{ s.} \tag{63}
\]

The ground point position and velocity at this instant are

\[
r_E = (3296533.996, -3606135.417, 4099699.375) \text{ m,} \tag{64}
\]

\[
V_E = (262.88727, 240.31733, 0.00000) \text{ m/s.} \tag{65}
\]

(The \(z\) axis is chosen parallel to earth’s rotation axis.) These data allow us to compute the vector \(d_E\) and to transform it into the LRF:

\[
d_E^L = (-194154.8809, 662230.5410, 173092.5238) \text{ m,} \tag{66}
\]
Using the relativistic composition law for velocities, we first transform the given ERF wind velocities to the ECI, and then to the ERF. This is so we can calculate the expected fractional frequency shift. The velocity of the wind relative to the lidar apparatus will be
\[ \mathbf{w} = (-1095.2220, -5447.7191, -4897.9803) \text{ m/s}. \]

With these simulated values we can compute the fractional frequency shift that should be observed from Eq. (4):
\[ \frac{\Delta f''}{f''} = 3.9783903 \times 10^{-5}, \]

and the quantity
\[ \frac{\Delta f''}{f''} = c(-1.9891555694 \times 10^{-5}). \]

The vector \( \mathbf{A} \) defined above is
\[ \mathbf{A} = (-0.272888029028, 0.93077643041, 0.24328452306). \]

Then the vector \( \mathbf{B} \) may be computed:
\[ \mathbf{B} = (-0.272888029028, 0.93077643041, 0.24328452306). \]

These two vectors turn out to be essentially equal; this is because the speed of the ground point is very small compared with the speed of light. In a first approximation one can in fact neglect the difference between \( \mathbf{B} \) and \( \mathbf{A} \). Resolving the vector \( \mathbf{B} \) into eastward, northward, and vertical components, we quote only the horizontal components since the wind is assumed to be horizontal:
\[ \mathbf{B} = (-0.50449226E, 0.79099629N). \]

These results can now be used to test Eq. (41). Using the input values of the wind at the ground point gives for the left-hand side of Eq. (41) 11.952212693 m/s, while evaluating the right-hand side gives 11.952212693 m/s. The agreement is extremely good, which gives one confidence that the simulation properly represents the theory.

It is of interest to examine the consequences of making approximations leading to Eq. (42). The LOS vector is defined in the ECI reference frame. The left-hand side of Eq. (42) has the nonrelativistic value 11.952212693 m/s, while the right-hand side evaluates to 11.950209 m/s. The difference is 2.4 parts in 10^4; there has been a loss of accuracy from the subtraction of two large terms on the right:

\[ \frac{\Delta f''}{f''} + \frac{c}{2 \frac{f''}{c}} (\mathbf{V}_L - \mathbf{V}_E) \cdot \mathbf{n}_t = (-5.963456998 \times 10^3 + 5.975359093 \times 10^3) \text{ m/s}. \]

Thus the lidar Doppler shift measurement is large because the spacecraft velocity is large. Most of the shift must be subtracted out to account for velocities of the spacecraft and ground point in order to measure the wind.

**B. Second Lidar Measurement**

Let a second measurement be made exactly 150 s later. For this measurement the lidar is 54.31° from the ground point zenith, whereas for the first measurement the lidar was 59.28° from the zenith. (These conditions were arbitrarily chosen and do not correspond to a constant scanning angle.) The physical configuration of the two lidar measurements is drawn to scale in Fig. 8; there it is seen that the first laser shot occurs as the lidar is approaching the scatterer, and the second shot occurs as the lidar is receding. During the time interval between the shots, the scatterer moves due to the combination of wind and earth rotation. Relativistic effects are too small to show up in Fig. 8.

For the second laser shot, the measured fractional frequency shift is
\[ \frac{\Delta f''}{f''} = -3.70437752 \times 10^{-5}, \]

and the result of evaluating the left side of Eq. (41) is -60.22015497 m/s, while evaluating the right-hand side gives -60.22015497 m/s. Again, the agreement is extremely good.

These two simulated measurements give the wind velocity relative to the ground point. For the first measure-
moment, the vector $\mathbf{B}$ has been given above. Assuming the wind has eastward and northward components $(w'_x, w'_n)$ gives us one equation:

$$-0.5044922625w'_x + 0.7909962943w'_n = 11.952212693 \text{ m/s}.$$  

(76)

For the second measurement, evaluation of the vector $\mathbf{B}$ gives

$$\mathbf{B} = (-0.4981724053, -0.33958394090, -0.797813889),$$  

(77)

and resolving this into eastward, northward, and vertical components accounting for the additional rotation of earth,

$$\mathbf{B} = (-0.3933068009E, -0.8628684348N).$$  

(78)

Assuming that the wind remains constant in between the measurements gives

$$-0.39330680097w'_x - 0.86286843481w'_n = -60.22015497 \text{ m/s}.$$  

(79)

Solving the two simultaneous equations (76) and (79) for the wind velocity gives

$$(w'_x, w'_n) = (50.000000, 46.999999) \text{ m/s},$$  

(80)

which agrees very well with the assumed wind velocity components $(w_x, w_n) = (50, 47)$ m/s. If the wind has a vertical component, a third measurement is needed to determine all three wind components.

6. ACCELERATED SPACECRAFT

We reanalyze the measurement from the point of view of the ECI frame, but we include acceleration of the spacecraft and the effect of earth’s gravitational potential on the transmitted and detected signal frequencies. The spacecraft’s acceleration $\mathbf{g}$ in earth’s gravity field changes the spacecraft position and velocity during the light travel time, so the detected frequency will be affected. For a light travel time of 8 ms, the spacecraft velocity change will be about 0.07 m/s$^{-1}$. If not accounted for the measured wind velocity could be in error by about half this, or 0.035 m/s$^{-1}$. To treat this effect it is most convenient to analyze the measurement in the ECI frame, allowing for different spacecraft velocities at the transmission and detection events. The calculation will be done without making any approximations in the relativistic transformations; then contributions of various orders will be discussed.

Let $t_s$ be the time of transmission of a wavefront from the lidar apparatus, and suppose it arrives at the scatterer at time $t_d$. At the instant of transmission the lidar has velocity $\mathbf{V}_L(t_s)$ and position $\mathbf{r}_L(t_s)$. The pulse is immediately scattered back into a detector, which is assumed to be coincident with the transmitter (monostatic lidar). The arrival time of the scattered wavefront at the detector is denoted by $t_d$. During the propagation delay time $t_d - t_s$ the detector velocity will change to

$$\mathbf{V}_L(t_d) = \mathbf{V}_L(t_s) + \mathbf{g}(t_d - t_s),$$  

(81)

and the detector position will change to

$$\mathbf{r}_L(t_d) = \mathbf{r}_L(t_s) + \mathbf{V}_L(t_s)(t_d - t_s) + \frac{1}{2} \mathbf{g}(t_d - t_s)^2,$$  

(82)

where $\mathbf{g}$ is the acceleration due to earth’s gravity at the spacecraft. We let $\mathbf{n}_s$ and $\mathbf{n}_d$ be unit vectors pointing along the path of the transmitted and scattered rays, respectively.

The interaction time of the radiation with the scatterer is assumed to be so short that the positions of the transmission event, scattering event, and detection event can be described as happening at $\mathbf{r}_L(t_s)$, $\mathbf{r}_s(t_d)$, and $\mathbf{r}_d(t_d)$, respectively. We neglect atmospheric refraction or dispersion effects and confine this discussion to relativistic effects. For the transmitted radiation, we consider two successive wavefronts that start at $t_s$ and $t_s + dt_s$ and arrive at the scatterer at times $t_s$ and $t_s + dt_s$, respectively. Then from the constancy of the speed of light we have

$$t_s - t_s = \frac{1}{c} |\mathbf{r}_s(t_s) - \mathbf{r}_s(t_s)|,$$  

(83)

and differentiating to obtain expressions for the differentials $dt_s$, $dt_s$, we have

$$dt_s - dt_s = \frac{1}{c} \left( (\mathbf{r}_s(t_s) - \mathbf{r}_s(t_s)) \cdot (\mathbf{dr}_s(t_s) - \mathbf{dr}_s(t_s)) \right) \left| \mathbf{r}_s(t_s) - \mathbf{r}_s(t_s) \right|.$$  

(84)

The unit vector along the LOS of the transmitted ray is

$$\mathbf{n}_s = \frac{\mathbf{r}_s(t_s) - \mathbf{r}_s(t_s)}{\left| \mathbf{r}_s(t_s) - \mathbf{r}_s(t_s) \right|}.$$  

(85)

Also, the position increments are related to the velocities

$$\mathbf{dr}_s = \frac{d\mathbf{r}_s}{dt_s} dt_s = \mathbf{w} dt_s, \quad \mathbf{dr}_s = \mathbf{V}_L(t_s) dt_s.$$  

(86)

Thus

$$dt_s - dt_s = \mathbf{n}_s \cdot (\mathbf{w} dt_s - \mathbf{V}_L(t_s) dt_s)/c,$$  

(87)

and this may be rearranged to give

$$dt_s \left( 1 - \frac{\mathbf{n}_s \cdot \mathbf{w}}{c} \right) = dt_s \left( \frac{\mathbf{n}_s \cdot \mathbf{V}_L(t_s)}{c} \right).$$  

(88)

The coordinate time interval $dt_s$ can be related to the corresponding proper time $\Delta \tau$ on a standard clock at the transmitter, and hence to the proper frequency of the transmitted signal, by using the ordinary scalar invariant of the gravitational field. We shall suppose that the proper time interval $\Delta \tau$ corresponds to the emission of two successive wavefronts by the transmitter so that $\Delta \tau = 1/f_0$. The proper time is given to sufficient accuracy by

$$(cd \Delta \tau)^2 = \left( 1 + \frac{2\Phi(r_s)}{c^4} \right) (cdt_s)^2 - (dx^2 + dy^2 + dz^2),$$  

(89)
Substituting this into Eq. (97),

\[ \Delta t_d = \frac{1}{f''} \sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2} - \Delta t_d}, \]

where \( \Phi(r_d) \) is the Newtonian gravitational potential of earth at the transmission event. Thus

\[ \Delta t^2 = \sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2}}. \]

For the emission of two successive wavefronts, this time is the reciprocal of the frequency, and so

\[ \Delta t_t = \frac{1}{f''} \sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2}}, \]

and therefore using Eq. (88)

\[ \Delta t_t \left( 1 - \frac{n_d \cdot w}{c} \right) = \frac{\left( 1 - \frac{n_t \cdot V_L(t_t)}{c} \right)}{f''} \sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2}}. \]

We perform a similar analysis for the successively scattered wavefronts coming back to the detector and use the result to eliminate \( \Delta t_t \). From constancy of the speed of light,

\[ t_d - t_s = \frac{1}{c} | \mathbf{r}_d(t_d) - \mathbf{r}_s(t_s) |, \]

and differentiating to obtain expressions for the differentials \( dt_d, dt_s \), we have

\[ dt_d - dt_s = \frac{1}{c} \left( (\mathbf{r}_d(t_d) - \mathbf{r}_s(t_s)) \cdot (d\mathbf{r}_d(t_d) - d\mathbf{r}_s(t_s)) \right). \]

The unit vector along the LOS of the scattered ray is

\[ \mathbf{n}_d = \frac{\mathbf{r}_d(t_d) - \mathbf{r}_s(t_s)}{| \mathbf{r}_d(t_d) - \mathbf{r}_s(t_s) |} \]

so inserting velocities of wind and lidar detector, substituting in the definition of \( \mathbf{n}_d \) and rearranging, we obtain

\[ \Delta t_t \left( 1 - \frac{n_d \cdot w}{c} \right) = \Delta t_d \left( 1 - \frac{n_d \cdot V_L(t_d)}{c} \right). \]

The coordinate time interval \( \Delta t_t \) at the detector is related to the proper time increment at the detector, and hence to the detected frequency \( f'' \), by

\[ \Delta t_t' = \frac{1}{f''} \sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2} - \Delta t_d}. \]

and therefore

\[ \Delta t_t' = \frac{1}{f''} \sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2} - \Delta t_d}. \]

Substituting this into Eq. (97),

\[ \frac{dt}{df} \left( 1 - \frac{n_d \cdot w}{c} \right) = \left( 1 - \frac{n_d \cdot V_L(t_d)}{c} \right) \left( 1 - \frac{n_d \cdot V_L(t_d)}{c} \right). \]

Dividing Eq. (100) into Eq. (93), canceling \( dt_t \), and rearranging, we obtain the following expression for the ratio of detected to transmitted frequency in the LRF:

\[ \frac{f''}{f_0''} = \frac{\sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2} (1 - n_d \cdot w/c) (1 - n_d \cdot V_L(t_d)/c)}}{\sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2} (1 - n_d \cdot w/c) (1 - n_d \cdot V_L(t_d)/c)}}. \]

This relationship contains all the information available to determine the wind velocity. It allows for the possibility that \( V_L(t_d) \) and \( r_d(t_d) \) differ from \( V_L(t_t) \) and \( r_d(t_t) \), respectively. The desired wind velocity \( \mathbf{w'} \) will be introduced by making a Lorentz transformation.

Let us first consider the ratio of square roots that occurs on the right side of Eq. (101). If the spacecraft orbit were perfectly circular, then to a high degree of approximation the spacecraft speed and altitude would not change and the square root factors would cancel. Even if the orbit is slightly eccentric, the eccentricity \( e \) is expected to be rather small, say \( e = 0.01 \). Accounting for the changes due to earth's gravity as in Eqs. (81) and (82), the ratio is approximately

\[ \frac{\sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2} (1 - n_d \cdot w/c) (1 - n_d \cdot V_L(t_d)/c)}}{\sqrt{1 + \frac{2\Phi(r_d)}{c^2} - \frac{V_L(t_d)^2}{c^2} (1 - n_d \cdot w/c) (1 - n_d \cdot V_L(t_d)/c)}} \approx 1 + 2 \frac{(t_d - t_t)}{c^2} \mathbf{V}_L \cdot \mathbf{g}. \]

and the last dot product will bring in a factor \( e \). Thus this term is small not only because the propagation delay interval is small but also because there is another factor of \( e \) and there are already factors \( c^{-2} \) in the denominator. The net effect is that under almost all circumstances the square root ratio differs from unity by less than \( 10^{-15} \) and can be neglected. We shall therefore not consider it further in this paper. The expression for the measured frequency reduces to

\[ \frac{f''}{f_0''} = \frac{\left( 1 - \frac{n_d \cdot w}{c} \right) (1 - n_d \cdot V_L(t_d)/c)}{\left( 1 - \frac{n_d \cdot w}{c} \right) (1 - n_d \cdot V_L(t_d)/c)}. \]

If Eq. (103) is transformed to the LRF, assuming the velocity \( \mathbf{V}_L \) is constant, the result can be shown to agree with Eq. (4). Thus Eq. (103) provides an alternative formulation, in terms of quantities specified in the ECI frame, from which the wind velocity may be derived.

With Eq. (103) we can estimate the principal contribution of spacecraft acceleration to the measurement. Introduce the fractional frequency shift by means of Eq. (5) and expand the right-hand side of Eq. (103), keeping only terms of first order in \( (w/c) \) and \( (V_L/c) \). Then
\[ 1 + \frac{\Delta f''}{f_0'} = 1 - \mathbf{n}_t \cdot \mathbf{w}/c + \mathbf{n}_d \cdot \mathbf{w}/c - \mathbf{n}_d \cdot \mathbf{V}_L(t_d)/c + \mathbf{n}_t \cdot \mathbf{V}_L(t_d)/c. \]  

(104)

To a very good approximation, \( \mathbf{n}_d = -\mathbf{n}_t \), and \( \mathbf{w} = \mathbf{V}_E + \mathbf{w}' \). Also, the acceleration causes a change in velocity given by Eq. (81). With a little rearrangement we obtain

\[ \mathbf{w}' \cdot \mathbf{n}_t = -\frac{\lambda''}{2} \frac{\Delta f''}{f_0'} + (\mathbf{V}_L - \mathbf{V}_E) \cdot \mathbf{n}_t + \frac{1}{2} \mathbf{n}_t \cdot \mathbf{g}(t_d - t_i). \]  

(105)

For reasonable distances of \( \approx 1260 \text{ km} \) between transmitter and target, the propagation delay is about \( (t_d - t_i) \approx 8.4 \times 10^{-3} \text{ s} \), and \( \mathbf{g} = 9 \text{ m s}^{-2} \). So the acceleration term in Eq. (105) could contribute to the wind speed as much as

\[ \frac{1}{2} \frac{\mathbf{n}_t \cdot \mathbf{g}(t_d - t_i)/c}{9 \times 8.4 \times 10^{-3} \text{ m s}^{-1}} = 0.04 \text{ m s}^{-1}, \]  

(106)

but this could be further reduced if the angle between \( \mathbf{n}_t \) and \( \mathbf{g} \) is large. We have already discussed the approximate cancellations of the two main terms on the right side of Eq. (105).

The next step in the calculation is to transform Eq. (103) so that the wind velocity \( \mathbf{w}' \) in the ERF appears explicitly, without making further approximations. The calculation is tedious, and to complete it with a minimum number of steps we shall use the Lorentz transformations in vector form. At any stage in the calculation one may choose whether to transform the LOS unit vectors \( \mathbf{n}_t \) and \( \mathbf{n}_d \) into some other reference frame. We shall use only parallel axis transformations from the ECI frame to one or the other of the LRF and ERF.

With this in mind, the starting point is the expression, Eq. (103). To save writing, we introduce the abbreviation

\[ F = \frac{\int_0^\prime \frac{\Delta f''}{f_0'}(1 - \mathbf{n}_t \cdot \mathbf{V}_L(t_d)/c})(1 - \mathbf{n}_d \cdot \mathbf{V}_L(t_d)/c). \]  

(107)

This quantity includes the measured frequency of the backscattered radiation and Doppler shift factors that can be calculated when the velocities and the LOSs are known in the ECI. Then Eq. (103) becomes

\[ F = \frac{(1 - \mathbf{n}_t \cdot \mathbf{w}/c)}{(1 - \mathbf{n}_d \cdot \mathbf{w}/c)c}, \]  

(108)

and solving for the terms involving \( \mathbf{w} \), we have

\[ \mathbf{w} \cdot (\mathbf{n}_t - \mathbf{F}_d) = c(1 - F). \]  

(109)

We next introduce the desired wind velocity \( \mathbf{w}' \) by substituting the velocity transformation, Eq. (19), into Eq. (109). Then several dot products involving \( \mathbf{w}' \) appear. The result can be written

\[ c(1 - F) \gamma_E \left( 1 + \frac{\mathbf{V}_E \cdot \mathbf{w}'}{c^2} \right) = \mathbf{w}' \cdot (\mathbf{n}_t - \mathbf{F}_d) + (\gamma_E - 1) \]

\[ \times \mathbf{w}' \cdot \mathbf{V}_E \mathbf{V}_E \cdot (\mathbf{n}_t - \mathbf{F}_d)/\mathbf{V}_E^2 + \gamma_E\mathbf{V}_E \cdot (\mathbf{n}_t - \mathbf{F}_d). \]  

(110)

We collect all terms involving \( \mathbf{w}' \) on the left:

\[ \mathbf{w}' \cdot \left( \mathbf{n}_t - \mathbf{F}_d \right) - \gamma_E(c(1 - F)) \frac{\mathbf{V}_E}{c^2} + (\gamma_E - 1) \frac{\mathbf{V}_E}{\mathbf{V}_E^2} \cdot \mathbf{V}_E \cdot (\mathbf{n}_t - \mathbf{F}_d). \]  

(111)

**Spacecraft Acceleration.** Let us again examine the effect of spacecraft acceleration by assuming the wind velocity is very small. The right side of Eq. (111) reduces to

\[ c(1 - \frac{\Delta f''}{f_0'}) \frac{\mathbf{n}_t \cdot \mathbf{V}_L(t_d) + \mathbf{n}_d \cdot \mathbf{V}_L(t_d)/c)}{\mathbf{V}_E \cdot (\mathbf{n}_t - \mathbf{n}_d)} = \mathbf{n}_t \cdot \mathbf{g}(t_d - t_i) \]

(112)

Inserting the spacecraft acceleration, from Eqs. (81) and (82), there will be a contribution to the wind speed of about

\[ \mathbf{n}_d \cdot \mathbf{g}(t_d - t_i)/c \approx 0.08 \text{ m s}^{-1}, \]  

(113)

as discussed in connection with Eq. (105). A factor of 1/2 arises from the left side of the equation, so the result agrees with Eq. (105). In the rest of this paper we shall ignore spacecraft acceleration and concentrate on other relativistic corrections.

**7. ALTERNATIVE FORMS FOR RELATIVISTIC CORRECTIONS**

Equation (111) is an alternative form of the result including relativistic corrections, with constant spacecraft velocity. This equation can be reduced to Eq. (41) by dividing each term in Eq. (111) by \( 2\gamma_E\gamma_L^2(1 - \mathbf{V}_E \cdot \mathbf{n}_t)/c(1 + \Delta f''/(2f_0')) \). For example, the following identities can be proved:

\[ 2\gamma_E^2 \left( 1 - \frac{\mathbf{V}_E \cdot \mathbf{n}_t}{c} \right) \left( 1 + \frac{1}{2} \frac{\Delta f''}{f_0'} \right) \mathbf{A} = \mathbf{n}_t - \mathbf{F}_d, \]

(114)

\[ 2\gamma_E \left( 1 - \frac{\mathbf{V}_E \cdot \mathbf{n}_t}{c} \right) \left( 1 + \frac{1}{2} \frac{\Delta f''}{f_0'} \right) \left( \frac{c}{2} \frac{\Delta f''}{f_0'} + \mathbf{V}_L \cdot \mathbf{n}_t \right) = c(1 - F). \]

(115)

In Eq. (111), the quantity into which \( \mathbf{w}' \) is dotted, as well as the quantity on the right side of Eq. (111), are expressed in terms of the measured frequency ratio \( \Delta f''/f_0' \) in the LRF and velocities specified in the ECI frame. Various cases of interest may be discussed; for example, relativistic corrections may be obtained by expanding Eq. (111) to the desired order in \( (\omega/c), (V_E/c), \) or \( (V_L/c) \).

Henceforward we shall neglect spacecraft acceleration and assume that

\[ \mathbf{V}_L(t_d) = \mathbf{V}_L(t_i). \]

(116)

In this case
Then putting this back in to Eq. (117) gives

\[ d_d + d_i = V_L (t_d - t_i), \quad (117) \]

and using \( c(t_d - t_i) = d_d + d_i \), one can exactly solve for \( d_d \) and \( d_d \) in terms of \( d_i \). The solutions are useful, so we quote them here. First, calculating the square of \( d_d \) gives rise to a quadratic equation for \( d_d \). By an argument similar to that leading to Eq. (27), the physical solution is

\[ d_d = d_i, \frac{1 - 2V_L \cdot n_i/c + V_L^2/c^2}{1 - V_L^2/c^2}. \quad (118) \]

Then putting this back in to Eq. (117) gives

\[ d_d = -d_i + 2 \frac{V_L d_i, 1 - V_L \cdot n_i/c}{1 - V_L^2/c^2}. \quad (119) \]

If all first-order corrections are neglected in Eq. (111), \( F = \frac{1 + \Delta f''}{f_0''} \), and the equation can be reduced to

\[ w' \cdot n_i = -c \frac{\Delta f''}{2 f_0''} + \frac{V_L \cdot n_i - V_E \cdot n_i}{1 - V_L^2/c^2}, \quad (120) \]

which is equivalent to Eq. (42).

To obtain first-order corrections, we first need to expand \( n_i \) in terms of other quantities. This comes from Eqs. (118) and (119) and is

\[ n_d = \frac{d_d}{d_d} = -n_i (1 + 2V_L \cdot n_i/c) + 2V_L/c. \quad (121) \]

Expanding Eq. (111) to leading order in \((w'/c), (V_E/c)\), and \((V_L/c)\), the following result is obtained:

\[ (w' + V_E - V_L) \cdot (n_i (1 + V_L \cdot n_i/c) - V_L) = -c \frac{\Delta f''}{2 f_0''} - 1 + \frac{\Delta f''}{2 f_0''}, \quad (122) \]

Equation (122) agrees with Eq. (45) to this order. Since Eq. (111) has the same physical content as Eq. (41), expansions to higher order will also agree.

8. CORRECTIONS NEGLECTING RELATIVISTIC ROTATION OF AXES

In this section we investigate relativistic corrections under the assumption that a direct transformation between LRF and ERF can be made, which means neglecting the difference in orientation between the two sets of axes. Equation (18) allows us to introduce the velocity \( V_L \) of the LRF relative to the point on the ground. Let the velocity of the lidar apparatus as measured in the ECI frame be \( V_L \). Substituting this for \( v \) in Eq. (18) gives

\[ V_L = \frac{\sqrt{1 - V_E^2/c^2} \cdot V_L}{c + (1 - \sqrt{1 - V_E^2/c^2}) \cdot V_E \cdot V_L / V_E c - V_E / c}. \quad (123) \]

Since the positions of spacecraft and earth point are presumed known, all quantities appearing on the right side of this equation are known. We may thus proceed to transform the observable quantity, Eq. (7), to the ERF. In the following equations, we make direct Lorentz transformations (recall quantities with primes are measured in the ERF): \( r' = r^\prime + (\gamma' - 1) \frac{V_L^\prime \cdot r^\prime}{V_L^2} + \gamma' \frac{V_L}{c} \cdot t^\prime \), \quad (124) \]

\[ c t' = \gamma' \left( c t^\prime + \frac{V_L \cdot r^\prime}{c} \right), \quad (125) \]

where \( \gamma' \) is a function of \( V_L \).

Transformation to the ERF. We derive an equation satisfied by the wind velocity in the ERF. To do this we transform the observable directly from the LRF into the ERF, using the ECI frame only as an intermediary for calculation of the relative velocity of the two frames. Using the Lorentz transformations, for the transmission event we have

\[ r'_t = r'' - t'' = (\gamma' - 1) \frac{V_L^t \cdot r''}{V_L^2} + \gamma' \frac{V_L^t}{c} (c t'' - c t'_t), \quad (127) \]

and for the arrival event of the pulse at the ground point,

\[ r'_s = r'' + (\gamma'' - 1) \frac{V_L^s \cdot r''}{V_L^2} + \gamma'' \frac{V_L^s}{c} (c t'' - c t'_s), \quad (128) \]

Subtracting the first of these equations from the second, we have an expression for the vector \( d'_t \) from the transmission event to the arrival event in the ERF:

\[ d'_t = r'_s - r'_t = d'' + (\gamma'' - 1) \frac{V_L^s \cdot d''}{V_L^2} + \gamma'' \frac{V_L^s}{c} (c t'' - c t'_t), \quad (129) \]

But \( c t'' - c t'_t = d''_t \) in the LRF, so we find for the path vectors

\[ d'_t = d'' + (\gamma'' - 1) \frac{V_L^s \cdot d''}{V_L^2} + \gamma'' \frac{V_L^s}{c} d''_t. \quad (130) \]

We can find the magnitude of \( d'_t \) by dotting it into itself. The result is

\[ d'_t = \gamma'' d''_t \left( 1 + \frac{V_L \cdot d''_t}{c d''_t} \right), \quad (131) \]

and the inverses are
\[ d_i' = \gamma_i d_i \left( 1 - \frac{V_i V_L}{c^2} \right), \quad (132) \]

\[ d_i'' = \gamma_i d_i \left( 1 - \frac{V_i V_L}{c^2} \right), \quad (133) \]

We leave the fractional frequency difference as measured in the LRF unchanged but transform the expression on the left side of Eq. (7). Then we need the quantity

\[ \frac{w'' \cdot d_i''}{c d_i''}, \quad (134) \]

The object of the following derivation is to simplify the above expression as much as possible, isolating all quantities involving \( w' \) in the numerator of one side of the equation. Note that everything is presumed known in this equation except \( w' \). Putting some of the factors that are in the denominator of Eq. (136) on the left, working out all the terms in the dot product and simplifying, we obtain

\[ \frac{w'' \cdot d_i''}{\gamma_i^2 \frac{1}{1 + \frac{1}{2} \frac{\Delta f''}{f_0'}}} = \frac{-c \Delta f''}{\gamma_i^2 \frac{1}{1 + \frac{1}{2} \frac{\Delta f''}{f_0'}}} \left( 1 - \frac{V_i \cdot w'}{c^2} \right) \]

\[ = \frac{w' \cdot d_i'}{c d_i'} = \gamma_i^2 \frac{-V_i \cdot d_i'}{c} + \frac{V_i^2}{d_i'} \left( 1 - \frac{V_i \cdot d_i'}{c} \right). \quad (137) \]

The terms involving \( w' \) can be put on one side of the equation:

\[ \frac{w' \cdot n_i'}{\gamma_i^2 \frac{1}{1 + \frac{1}{2} \frac{\Delta f''}{f_0'}}} = \frac{-c \Delta f''}{\gamma_i^2 \frac{1}{1 + \frac{1}{2} \frac{\Delta f''}{f_0'}}} \left( 1 - \frac{V_i \cdot n_i'}{c^2} \right) \]

\[ = \frac{-c \Delta f''}{\gamma_i^2 \frac{1}{1 + \frac{1}{2} \frac{\Delta f''}{f_0'}}} \left( 1 - \frac{V_i \cdot n_i'}{c} \right), \quad (138) \]

where \( n_i' = d_i'/d_i'' \). The quantities appearing in this expression should be known, either from measurement or from knowledge of the spacecraft orbit and ground position given the time of the measurement, except the wind velocity \( w' \) in the ERF. Let us introduce a vector \( D_i' \) by means of

\[ D_i' = \frac{n_i'}{\gamma_i^2 \frac{1}{1 + \frac{1}{2} \frac{\Delta f''}{f_0'}}} - \frac{V_i \cdot n_i'}{c^2} \left( 1 - \frac{V_i \cdot n_i'}{c} \right). \quad (139) \]

Then the result for the wind velocity in the ERF involves a dot product:

\[ w' \cdot D_i' = \frac{-c \Delta f''}{\gamma_i^2 \frac{1}{1 + \frac{1}{2} \frac{\Delta f''}{f_0'}}} \left( 1 - \frac{V_i \cdot n_i'}{c} \right). \quad (140) \]

The result gives the component of \( w' \), the wind velocity, in the ERF, along \( D_i' \). \( D_i' \) is not quite a unit vector. It also depends on the measured frequency shift, so it is a complicated but presumably known object. Relativistic corrections of all orders are contained in this result. For example, there are several relativistic corrections contained in the expression for the relative velocity \( V_i \), given in Eq. (123). However, because the direct transformation has ignored the nonparallelism of the ERF and LRF axes, the results cannot be expected to be accurate to \( O(c^{-2}) \). Expanding Eq. (140) to first order in \( (V_i/c) \), the result can be shown to agree with Eq. (45).
9. NONPARALLELISM OF AXES IN ERF AND LRF

The theory presented in the previous sections treated the LRF, the ECI frame, and the ERF as though they were parallel. This is not strictly true, since it is well known that when two noncollinear boosts are involved the net transformation involves a boost and a rotation. In the present case, the velocities \( \mathbf{V}_E \) and \( \mathbf{V}_L \) are generally not in the same direction. Therefore the axes of the LRF and the ERF are generally not parallel. Equation (123) gives \( \mathbf{V}_L \), the velocity of the LRF relative to the ERF. The velocity of the ERF as measured in the LRF would be of the same form as Eq. (123) with \( \mathbf{V}_E \) and \( \mathbf{V}_L \) interchanged. We call this velocity \( \mathbf{V}_E' \), and find

\[
\mathbf{V}_E' = \frac{\mathbf{V}_E - \mathbf{V}_L}{c}.
\]

It is easy to see that \( \mathbf{V}_L' \) and \( \mathbf{V}_E' \) have the same magnitude, but they do not point in opposite directions in the vector sense. If we let \( \phi \) be the angle required to rotate \( \mathbf{V}_L' \) into \( \mathbf{V}_E' \), and let \( \hat{\mathbf{N}} \) be a unit vector along the rotation axis, then

\[
\hat{\mathbf{N}} \sin(\phi) = \frac{\mathbf{V}_L' \times (-\mathbf{V}_E')}{V_L'^2}.
\]

After some algebra, and reducing the expression to its lowest-order contribution by expanding in powers of \( (V_L/c) \) and \( (V_E/c) \), we find

\[
\hat{\mathbf{N}} \sin(\phi) = \frac{\mathbf{V}_L \times (\mathbf{V}_E)}{2c^2}.
\]

This rotation angle is typically only a few parts in \( 10^{11} \). This correction is of order \( (V_L/c) \) or \( (V_E/c) \) smaller than the first-order relativistic corrections discussed earlier. It may be negligible in the current application.

Using the transformations developed in this paper, and neglecting the difference in direction of \( \mathbf{V}_E' \) and \( \mathbf{V}_L' \), from Eq. (140) one may transform directly from the ERF to the LRF and expand to second order and obtain

\[
\mathbf{w'} = \left( \mathbf{n'}_L + \frac{\mathbf{V}_L'}{c} \left( 1 - \frac{\mathbf{V}_L' \cdot \mathbf{n'}_L}{c} \right) \right) - \frac{c \Delta \phi''}{2 f_0} \left( 1 - \frac{\mathbf{V}_L' \cdot \mathbf{n'}_L}{c} \right)
\]

\[
- \frac{c \Delta \phi''}{2 f_0} \left( 1 + \frac{\mathbf{V}_L'^2}{2c^2} \right) \left( \mathbf{V}_L' \cdot \mathbf{n'}_L - \frac{\mathbf{V}_L'^2}{c} \right).
\]

On the other hand, one can transform directly from the LRF to the ERF using Eq. (7), assuming the velocity of the LRF relative to the ERF is \( \mathbf{V}_E' \), expand to second order and obtain

\[
\mathbf{w'} = \left( \mathbf{n'}_E - \frac{\mathbf{V}_E'}{c} \left( 1 + \frac{\mathbf{V}_E' \cdot \mathbf{n'}_E}{c} \right) \right) - \frac{c \Delta \phi''}{2 f_0} \left( 1 + \frac{\mathbf{V}_E'^2}{2c^2} + \frac{\mathbf{V}_E' \cdot \mathbf{n'}_L}{c} \right)
\]

\[
- \frac{c \Delta \phi''}{2 f_0} \left( 1 + \frac{\mathbf{V}_E'^2}{2c^2} - \frac{\mathbf{V}_E \cdot \mathbf{n'}_L}{c} \right) - \frac{c \Delta \phi''}{2 f_0} \left( 1 + \frac{\mathbf{V}_E'^2}{2c^2} + \frac{\mathbf{V}_E' \cdot \mathbf{n'}_E}{c} \right)
\]

Equations (144) and (145) appear at first to be identical, assuming \( \mathbf{V}_L' = -\mathbf{V}_E' \); however, such an identity ignores nonparallelism of axes of the ERF and LRF. Neither Eq. (144) nor Eq. (145) agrees with results derived earlier [Eqs. (41), (111), and (140)] that were derived without assuming the axes of the ERF and LRF were parallel; thus they are incorrect and should not be used beyond first order in \( (v/c) \), where \( v \) is any velocity in the paper. Eqs. (144) and (145) provide alternative formulations, depending on different velocity variables, that are valid only to first order in \( (v/c) \). The difference of orientation of the axes in the LRF and in the ERF is illustrated in Fig. 9, in which the vectors \( \mathbf{V}_L' \) and \( \mathbf{V}_E' \) are drawn in the plane of the paper. These two vectors would be antiparallel if the reference frame axes were parallel. The angle between \( \mathbf{V}_L' \) and \( \mathbf{V}_E' \) is expressed approximately in Eq. (143) in terms of a vector rotation angle \( \mathbf{V}_L' \times \mathbf{V}_E'/2c^2 \), which would be normal to the plane of the paper.

10. ON-BOARD GPS RECEIVER

If a GPS receiver is placed on the spacecraft, then the lidar position, velocity, and time may be available directly from the receiver. Ordinarily the GPS receiver will provide position and time in earth-centered earth-fixed (ECEF) coordinates. (The origins of the ECEF frame and ERF are at earth’s center and at the ground point, respectively.) The ECEF frame is rotating with respect to the ECI frame; the rotation introduces some new relativistic
effects. We show here how to transform measurements made by the GPS receiver to the ECI frame so that the previous analysis then applies. Fortunately the rotational speed is so slow that only effects of first order in \((v/c)\) need to be considered. The ECEF frame must be distinguished from the ERF that we have introduced because of the time delay between transmission and detection. In the noninertial ECEF frame light does not travel in all directions with a unique speed \(c\).

In the present section we shall place primes on quantities measured in the rotating frame, keeping in mind that it is not an inertial frame. Let us imagine freezing the axes of the ECEF frame at the instant a signal pulse is transmitted, and identifying the directions of these axes with those of an ECI frame. GPS time is effectively the time delay between transmission and detection. In the previous analysis then applies. Fortunately the rotational problems were developed and examples were given, demonstrating the consistency of the approach. Relativistic corrections were derived; for most purposes only the leading relativistic corrections need be considered in converting lidar Doppler measurements to wind velocities, but the theory given here [as in Eq. (101)] should be good to all orders in the small parameters \((v/c)\), where \(v\) is any velocity in the problem.

### 11. SUMMARY

In this paper derivations of the Doppler shift measured by a lidar apparatus, when the incident beam is backscattered from an aerosol particle carried along in the wind, have been presented. One of these derivations was from the point of view of the rest frame of the lidar [see Section 4, Eq. (41)], and the other was from the point of view of the ECI frame and permitted the calculation of gravitational effects [see Section 6, Eq. (101)]. The results agree when acceleration and gravitational frequency shifts are neglected, and show how relativistic corrections enter into the fundamental equation for the wind velocity in the lidar rest frame. Expressions for the wind velocity in the rest frame of the ground point, the earth-fixed point underneath the wind, were derived. Several ways of defining the LOS vector were discussed, and relationships between them were derived. Simulations of the lidar measurement were developed and examples were given, demonstrating the consistency of the approach. Relativistic corrections were derived; for most purposes only the leading relativistic corrections need be considered in converting lidar Doppler measurements to wind velocities, but the theory given here [as in Eq. (101)] should be good to all orders in the small parameters \((v/c)\), where \(v\) is any velocity in the problem.

### ACKNOWLEDGMENT

I am grateful to Michael Kavaya for suggesting this investigation and for many helpful discussions.
REFERENCES