Fourier Series

The Fourier series is a mathematical method to convert a function in the "Amplitude vs Time Domain" to the "Amplitude vs Frequency Domain" for ***periodic*** function.

Fourier Transform

The Fourier series is a mathematical method to convert a function in the "Amplitude vs Time Domain" to the "Amplitude vs Frequency Domain" for ***non – periodic*** function.

Background of Fourier Series Representation

The French mathematician Fourier found that any periodic waveform, that is, a waveform that repeats itself after some time, can be expressed as a series of harmonically related sinusoids, i.e., sinusoids whose frequencies are multiples of a fundamental frequency (or first harmonic). For example, a series of sinusoids with frequencies $1Hz, 2Hz, 3Hz$ and so on, contains the fundamental frequency of $1Hz$, a second harmonic of $2Hz$, a third harmonic of $3Hz$, and so on. In general, any periodic waveform can be expressed as

$$f(t) = \frac{1}{2}a_0 + a_1\cos\omega t + a_2\cos 2\omega t + a_3\cos 3\omega t + \ldots\ldots\ldots + b_1\sin\omega t + b_2\sin 2\omega t + b_3\sin 3\omega t + \ldots\ldots\ldots \quad \textbf{(01)}$$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t) \quad \textbf{(02)}$$

Here,

$\frac{a_0}{2}$ = Constant = Represents the DC (average) value of $f(t)$

a_n = Coefficient. b_n = Coefficient.

We need to find a_0, a_n & b_n to write Fourier series for any periodic function.

Evaluation of the Coefficients

We all know the following trigonometric relations:

$$\int_0^{2\pi} \sin(mt)dt = 0 \quad \textbf{(03)}$$

$$\int_0^{2\pi} \cos(nt)dt = 0 \quad \textbf{(04)}$$

$$\int_0^{2\pi} \sin(mt)\times\cos(nt)dt = 0 \quad \textbf{(05)}$$

Now, If m and n are different integers, then

$$\int_0^{2\pi} \sin(mt)\times\sin(nt)dt = 0 \qquad \textbf{(06)}$$

$$\int_0^{2\pi} \cos(mt)\times\cos(nt)dt = 0 \qquad \textbf{(07)}$$

However, if $m = n$

$$\int_0^{2\pi} \sin(mt)\times\sin(nt)dt = \int_0^{2\pi} \sin(mt)\times\sin(mt)dt = \int_0^{2\pi} \sin^2(mt)dt = \frac{1}{2}\int_0^{2\pi} 2\sin^2(mt)dt$$
$$= \frac{1}{2}\int_0^{2\pi} [1-\cos(2mt)]dt = \pi \qquad \textbf{(08)}$$

$$\int_0^{2\pi} \cos(mt)\times\cos(nt)dt = \int_0^{2\pi} \cos(mt)\times\cos(mt)dt = \int_0^{2\pi} \cos^2(mt)dt = \frac{1}{2}\int_0^{2\pi} 2\cos^2(mt)dt$$
$$= \frac{1}{2}\int_0^{2\pi} [1+2\cos(2mt)]dt = \pi \qquad \textbf{(09)}$$

Now, if I want to find coefficient a_2 in equation (01), let's multiply the equation (01) by $\cos 2\omega t$ and integrate it from 0 to 2π

$$\int_0^{2\pi} f(t)\cos 2\omega t dt = \frac{1}{2}\int_0^{2\pi} a_0 \cos 2\omega t dt + \int_0^{2\pi} a_1 \cos\omega t \times \cos 2\omega t dt + \int_0^{2\pi} a_2 \cos 2\omega t \times \cos 2\omega t dt + \dots\dots$$
$$+ \int_0^{2\pi} b_1 \sin\omega t \times \cos 2\omega t dt + \int_0^{2\pi} b_2 \sin 2\omega t \times \cos 2\omega t dt + \int_0^{2\pi} b_3 \sin 3\omega t \times \cos 2\omega t dt + \dots\dots \qquad \textbf{(10)}$$

Using trigonometric identities mentioned in equations (03) to (09), we can write equation (10) as follows:

$$\int_0^{2\pi} f(t)\cos 2\omega t dt = 0 + 0 + a_2 \int_0^{2\pi} \cos^2(2\omega t)dt + \dots\dots + 0 + 0 + 0 + \dots.$$
$$\int_0^{2\pi} f(t)\cos 2\omega t dt = a_2 \int_0^{2\pi} \cos^2(2\omega t)dt$$
$$\int_0^{2\pi} f(t)\cos 2\omega t dt = a_2 \times \pi$$
$$\therefore a_2 = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos 2\omega t dt$$

This means we can find any coefficient a_n using the following formula:

$$\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(n\omega t)dt = \frac{2}{T}\int_0^{T} f(t)\cos(\frac{2\pi n t}{T})dt} \qquad \textbf{(11)}$$

Now, if I want to find coefficient b_2 in equation (01), let's multiply the equation (01) by $\sin 2\omega t$ and integrate it from 0 to 2π

$$\int_0^{2\pi} f(t)\sin 2\omega t dt = \frac{1}{2}\int_0^{2\pi} a_0 \sin 2\omega t dt + \int_0^{2\pi} a_1 \cos\omega t \times \sin 2\omega t dt + \int_0^{2\pi} a_2 \cos 2\omega t \times \sin 2\omega t dt + ...$$
$$+ \int_0^{2\pi} b_1 \sin\omega t \times \sin 2\omega t dt + \int_0^{2\pi} b_2 \sin 2\omega t \times \sin 2\omega t dt + \int_0^{2\pi} b_3 \sin 3\omega t \times \sin 2\omega t dt + ... \quad \textbf{(12)}$$

Using trigonometric identities mentioned in equations (03) to (09), we can write equation (12) as follows:

$$\int_0^{2\pi} f(t)\sin 2\omega t dt = 0 + 0 + 0 + ... + 0 + b_2 \int_0^{2\pi} \sin 2\omega t \times \sin 2\omega t dt + 0 + ...$$
$$\int_0^{2\pi} f(t)\sin 2\omega t dt = b_2 \int_0^{2\pi} \sin^2(2\omega t) dt$$
$$\int_0^{2\pi} f(t)\sin 2\omega t dt = b_2 \times \pi$$
$$\therefore b_2 = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin 2\omega t dt$$

This means we can find any coefficient b_n using the following formula:

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(n\omega t)dt = \frac{2}{T}\int_0^{T} f(t)\sin(\frac{2\pi nt}{T})dt} \quad \textbf{(13)}$$

Now, if I want to find coefficient a_0 in equation (01), let's simply integrate equation (01) from 0 to 2π

$$\int_0^{2\pi} f(t)dt = \frac{1}{2}a_0 \int_0^{2\pi} dt + \int_0^{2\pi} a_1 \cos\omega t dt + \int_0^{2\pi} a_2 \cos 2\omega t dt + ... + \int_0^{2\pi} b_1 \sin\omega t dt + \int_0^{2\pi} b_2 \sin 2\omega t dt + \int_0^{2\pi} b_3 \sin 3\omega t dt + ... \quad \textbf{(14)}$$

Using trigonometric identities mentioned in equations (03) to (09), we can write equation (14) as follows:

$$\int_0^{2\pi} f(t)dt = (\frac{1}{2}a_0 \times 2\pi) + 0 + 0 + ... + 0 + 0 + 0 + ...$$
$$\int_0^{2\pi} f(t)dt = (\frac{1}{2}a_0 \times 2\pi)$$
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)dt$$

This means we can find DC value or average value a_0 using the following formula:

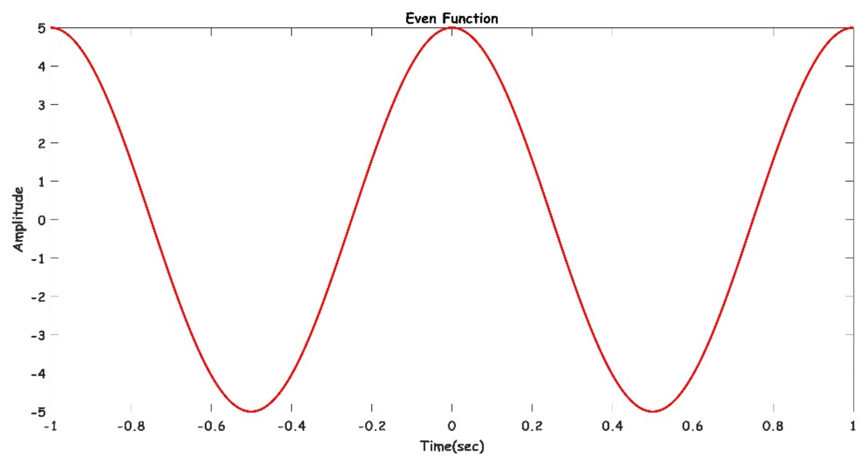
$$\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)dt} \quad \textbf{(15)}$$

Symmetry

Some waveforms have cosine terms only, while others have sine terms only. Still other waveforms have or have not DC or average components. Fortunately, it is possible to predict which terms will be present in the trigonometric Fourier series, by observing whether or not the given waveform possesses some kind of symmetry.

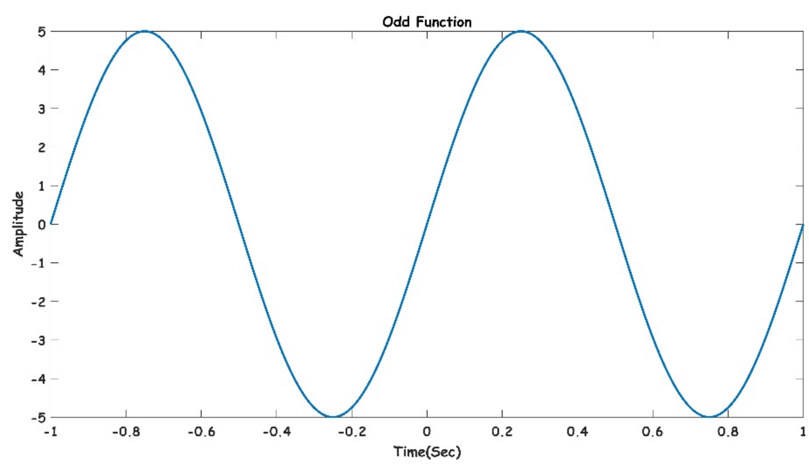
Even Symmetry

We know even functions are those for which $f(-t) = f(t)$. If a waveform has even symmetry, that is, if it is an even function, the series will consist of cosine terms only (a_n coefficients only), and a_0 may or may not be zero. In other words, if $f(t)$ is an even function, all the b_n coefficients will be zero.



Odd Symmetry

We know odd functions are those for which $f(t) = -f(-t)$. If a waveform has odd symmetry, that is, if it is an odd function, the series will consist of sin terms only (b_n coefficients only). In other words, if $f(t)$ is an odd function, all the a_n coefficients including a_0 will be zero.



Half – Wave Symmetry

A periodic waveform with period T has half – wave symmetry if $f(t) = -f(t \pm \frac{T}{2})$. If a waveform has half – wave symmetry, the series will consist all odd harmonics only (odd sin and odd cosine terms, i.e. odd a_n and odd b_n). All even harmonics will be zero.

Exponential Form of the Fourier Series

The Fourier series are often expressed in exponential form. The advantage of the exponential form is that we only need to perform one integration rather than two. Moreover, in most cases the integration is simpler. The exponential form is derived from the trigonometric form by substitution of

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{j2}$$

So, we can write equation (01) as follows:

$$f(t) = \frac{1}{2}a_0 + a_1(\frac{e^{j\omega t} + e^{-j\omega t}}{2}) + a_2(\frac{e^{j2\omega t} + e^{-j2\omega t}}{2}) + a_3(\frac{e^{j3\omega t} + e^{-j3\omega t}}{2}) + \dots + b_1(\frac{e^{j\omega t} - e^{-j\omega t}}{j2}) + b_2(\frac{e^{j2\omega t} - e^{-j2\omega t}}{j2}) + b_3(\frac{e^{j3\omega t} - e^{-j3\omega t}}{j2}) + \dots \quad \textbf{(16)}$$

Grouping terms with same exponents in equation (16) we get,

$$f(t) = \dots + (\frac{a_3}{2} - \frac{b_3}{j2})e^{-j3\omega t} + (\frac{a_2}{2} - \frac{b_2}{j2})e^{-j2\omega t} + (\frac{a_1}{2} - \frac{b_1}{j2})e^{-j\omega t} + \frac{1}{2}a_0 + (\frac{a_1}{2} + \frac{b_1}{j2})e^{j\omega t} + (\frac{a_2}{2} + \frac{b_2}{j2})e^{j2\omega t} + (\frac{a_3}{2} + \frac{b_3}{j2})e^{j3\omega t} + \dots \quad \textbf{(17)}$$

Now, let's define three (03) new unknown parameters as follows:

$$C_{-n} = (\frac{a_n}{2} - \frac{b_n}{j2}) = \frac{1}{2}(a_n - \frac{b_n}{j}) = \frac{1}{2}(a_n + jb_n)$$

$$C_n = (\frac{a_n}{2} + \frac{b_n}{j2}) = \frac{1}{2}(a_n + \frac{b_n}{j}) = \frac{1}{2}(a_n - jb_n)$$

$$C_0 = \frac{1}{2}a_0$$

Here, $C_{-n} = C_n^*$. They are complex conjugates

So, from equation (17) we can write,

$$f(t) = \dots + C_{-3}e^{-j3\omega t} + C_{-2}e^{-j2\omega t} + C_{-1}e^{-j\omega t} + C_0 + C_1e^{j\omega t} + C_2e^{j2\omega t} + C_3e^{j3\omega t} + \dots \quad \textbf{(18)}$$

Now, it's clearly visible the advantage of exponential Fourier series. As, $C_{-n} = C_n^*$, you just need to find out C_n using integration and then you can easily find C_{-n} without integration. So, this reduces mathematical complexity. Let's say, I want to find C_2. Now, multiplying equation (18) by $e^{-j2\omega t}$ and integrating over one period we find,

$$\int_0^{2\pi} f(t)e^{-j2\omega t}dt = \ldots + C_{-3}\int_0^{2\pi} e^{-j3\omega t}e^{-j2\omega t}dt + C_{-2}\int_0^{2\pi} e^{-j2\omega t}e^{-j2\omega t}dt + C_{-1}\int_0^{2\pi} e^{-j\omega t}e^{-j2\omega t}dt + C_0\int_0^{2\pi} e^{-j2\omega t}dt + C_1\int_0^{2\pi} e^{j\omega t}e^{-j2\omega t}dt + C_2\int_0^{2\pi} e^{j2\omega t}e^{-j2\omega t}dt + C_3\int_0^{2\pi} e^{j3\omega t}e^{-j2\omega t}dt + \ldots$$

Here in the above equation all the integrals are zero except $C_2\int_0^{2\pi} e^{j2\omega t}e^{-j2\omega t}dt$. So, we can write,

$$\int_0^{2\pi} f(t)e^{-j2\omega t}dt = \ldots + 0 + 0 + 0 + 0 + 0 + C_2\int_0^{2\pi} e^{j2\omega t}e^{-j2\omega t}dt + 0 + \ldots$$

$$\int_0^{2\pi} f(t)e^{-j2\omega t}dt = C_2\int_0^{2\pi} e^{j2\omega t}e^{-j2\omega t}dt$$

$$\int_0^{2\pi} f(t)e^{-j2\omega t}dt = C_2\int_0^{2\pi} dt$$

$$\int_0^{2\pi} f(t)e^{-j2\omega t}dt = 2\pi C_2$$

$$C_2 = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-j2\omega t}dt$$

This means we can find any coefficient C_n using the following formula:

$$\boxed{C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jn\omega t}dt = \frac{1}{T}\int_0^{T} f(t)e^{-jn\omega t}dt} \quad \textbf{(19)}$$

As you can calculate C_n using formula given in (19), you easily can find C_{-n} using the fact $C_{-n} = C_n^*$

Symmetry in Exponential Fourier Series

1. $C_{-n} = C_n^*$ always true here.
2. For even functions, all coefficients C_n and C_{-n} are real.

   Since, even functions have no sin terms (b_n are zero), we can write C_n and C_{-n} are as follows:

$$C_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{2}(a_n + j0) = \frac{1}{2}a_n$$

$$C_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}(a_n - j0) = \frac{1}{2}a_n$$

$$\therefore C_{-n} = C_n \quad \textbf{For Even Function}$$

3. For odd functions, all coefficients C_n and C_{-n} are imaginary.

   Since, odd functions have no cosine terms (a_n are zero), we can write C_n and C_{-n} are as follows:

$$C_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{2}(0 + jb_n) = \frac{1}{2}jb_n$$

$$C_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}(0 - jb_n) = -\frac{1}{2}jb_n$$

For Odd Function

4. If there is half – wave symmetry, $C_n = C_{-n} = 0$ for $n = even$

   We recall from the trigonometric Fourier series that if there is half-wave symmetry, all even harmonics are zero. Therefore, the coefficients a_n and b_n are both zero for $n = even$. Thus, $C_n = C_{-n} = 0$ for $n = even$.

Computation of RMS value from Fourier Series

We recall that the RMS value of a function, such as current $i(t)$ is defined as

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T i^2 dt}$$

If that current is represented in Fourier series, we can calculate the RMS value form the following formula:

$$I_{RMS} = \sqrt{I_0^{\,2} + \frac{1}{2}I_{1m}^{\,2} + \frac{1}{2}I_{2m}^{\,2} + \frac{1}{2}I_{3m}^{\,2} + \ldots.. + \frac{1}{2}I_{Nm}^{\,2}}$$

Lab Assignment 1 Solution

1. It's a periodic function.

2. This function follows:

   $f(t) = -f(-t)$ and $f(t) = -f(t \pm \frac{T}{2})$.

   So, it has both odd and half wave symmetry.

3. As this function follows odd symmetry, all a_n are zero including a_0 are zero. It will only have b_n coefficients. However, as this one also follows half – wave symmetry, only odd b_n coefficients ($b_1, b_3, b_5, ...etc$) will have numeric values. Others will be zero.

4. $$f(t) = \begin{cases} A; & 0 \le t \le \frac{T}{2} \\ -A; & \frac{T}{2} \le t \le T \end{cases}$$

5. Calculating coefficients: Using the symmetry conditions we can write

   $$a_0 = 0$$
   $$a_n = 0$$

   We have to calculate b_n. Using equation (13)

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(\frac{2\pi nt}{T})dt$$

$$b_n = \frac{2}{T}[\int_0^{T/2} A\sin(\frac{2\pi nt}{T})dt + \int_{T/2}^{T} -A\sin(\frac{2\pi nt}{T})dt$$

$$b_n = \frac{2}{T}[A \times \{-\cos(\frac{2\pi nt}{T})\}_0^{T/2} + A \times \{\cos(\frac{2\pi nt}{T})\}_{T/2}^{T}]$$

$$b_n = \frac{2}{T} \times \frac{T}{2\pi n} \times A[\{-\cos(\pi n) + 1 + \cos(2\pi n) - \cos(\pi n)]$$

$$b_n = \frac{A}{\pi n}[1 - 2\cos(\pi n) + \cos(2\pi n)]$$

If $n = even$

$$b_n = \frac{A}{\pi n}[1 - 2 + 1] = 0$$

$n = odd$

$$b_n = \frac{A}{\pi n}[1 + 2 + 1] = \frac{4A}{\pi n}$$

$$\therefore b_1 = \frac{4A}{\pi}$$

$$\therefore b_3 = \frac{4A}{3\pi}$$

$$\therefore b_5 = \frac{4A}{5\pi}$$

$$\vdots$$

6. The Fourier Series representation is then,

$$f(t) = (\frac{4A}{\pi}\sin\omega t + \frac{4A}{3\pi}\sin 3\omega t + \frac{4A}{5\pi}\sin 5\omega t + ...)$$

$$f(t) = \frac{4A}{\pi}(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + ...)$$

$$f(t) = \frac{4A}{\pi}\sum_{n=odd}\frac{1}{n}\sin n\omega t$$