# A Parallel Linear Solver for Block Circulant Linear Systems with Applications to Acoustics

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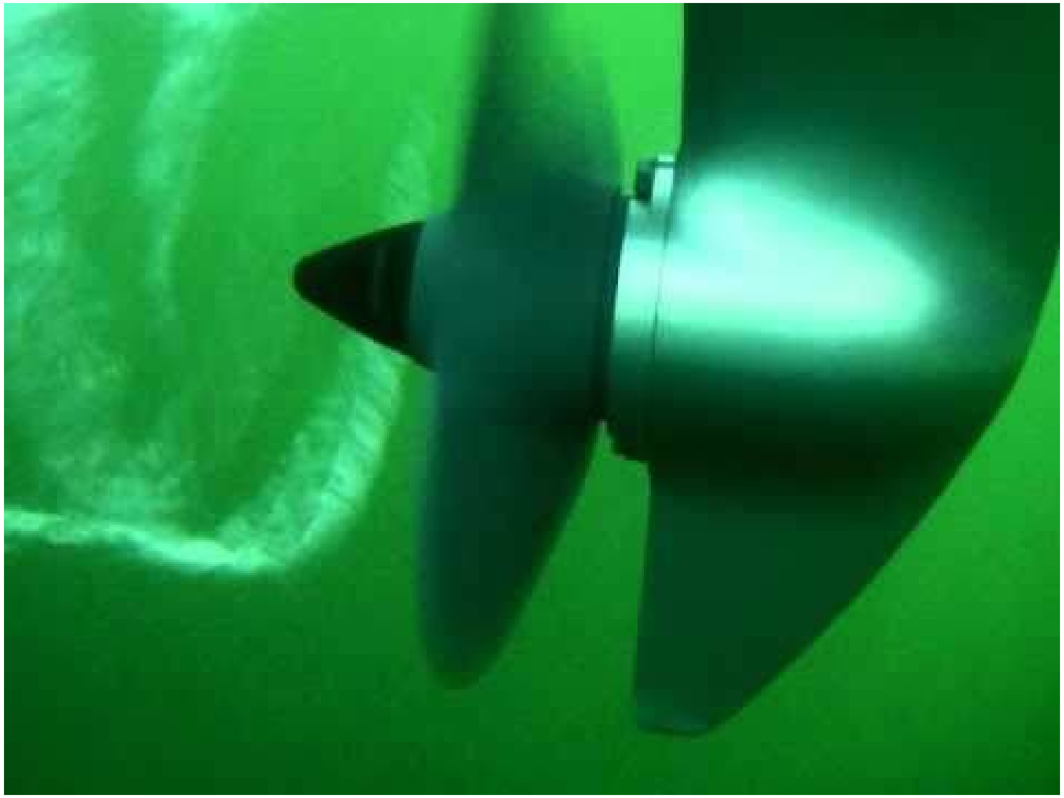
# MOTIVATION

# Application:  Vibrating Structures Immersed in Fluids



**Example:  Ships**

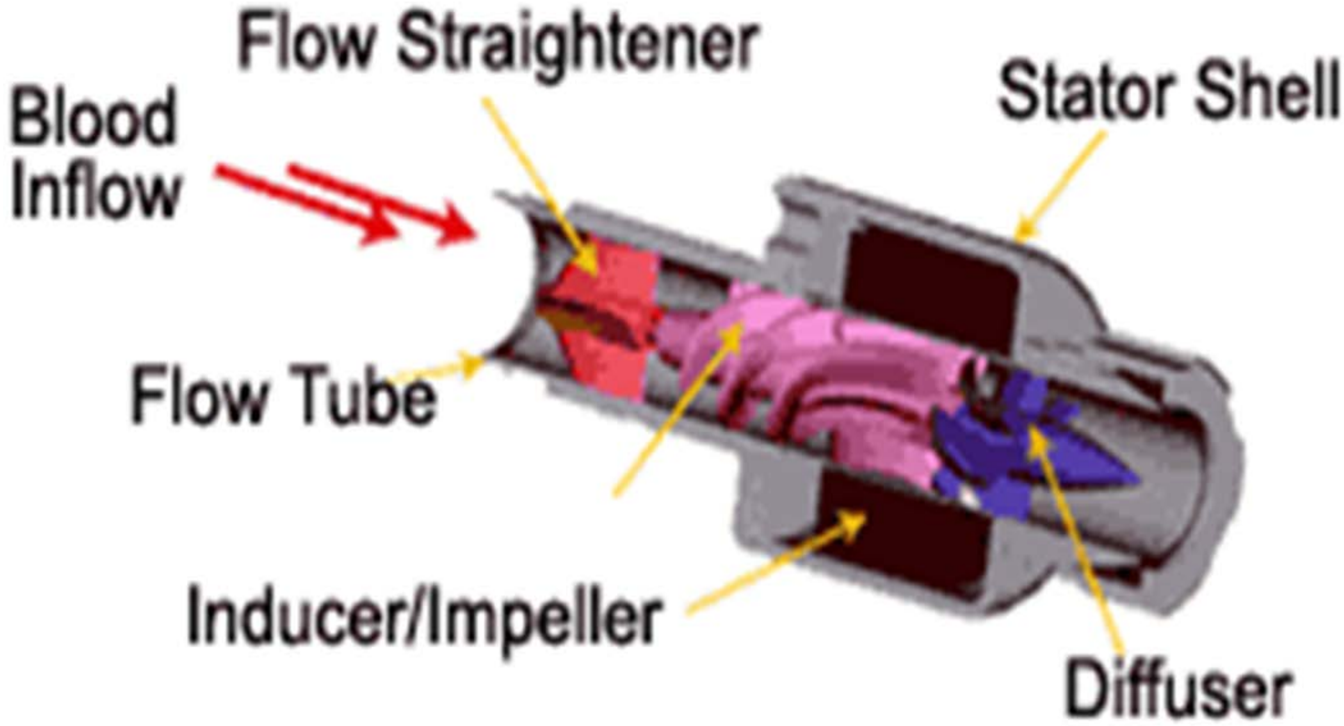
# Application:  Vibrating Structures Immersed in Fluids

**Example:  UAVs**

# Application:  Vibrating Structures Immersed in Fluids

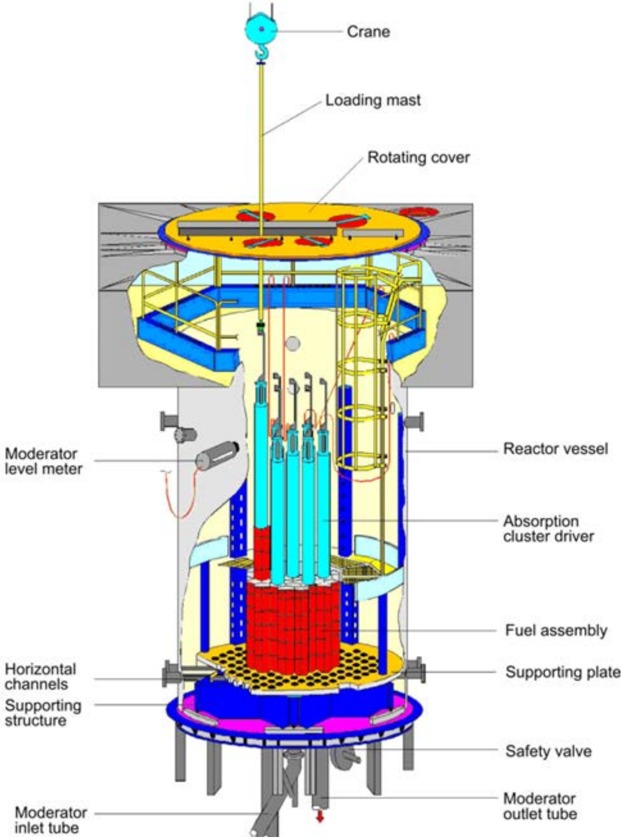
**Example:  Planes**

# Application: Vibrating Structures Immersed in Fluids



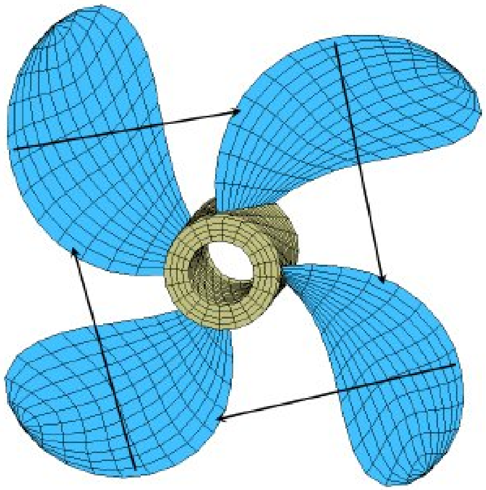

**Example: Blood Pumps**

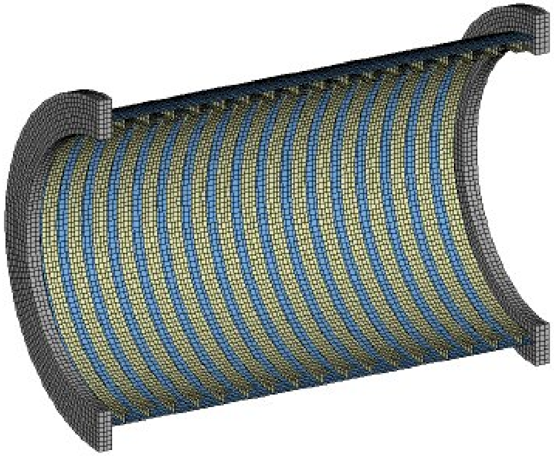
# Application: Vibrating Structures Immersed in Fluids




**Example: Reactors**

# Examples of Rotationally Symmetric Boundary Surfaces





**Real-world applications:** propellers, wind turbines, etcetera

# THE PROBLEM

# The Problem

**Goal:** To compute the acoustic radiation for a vibrating structure immersed in a fluid.

**Our focus:** Structures with rotationally symmetric boundary surfaces

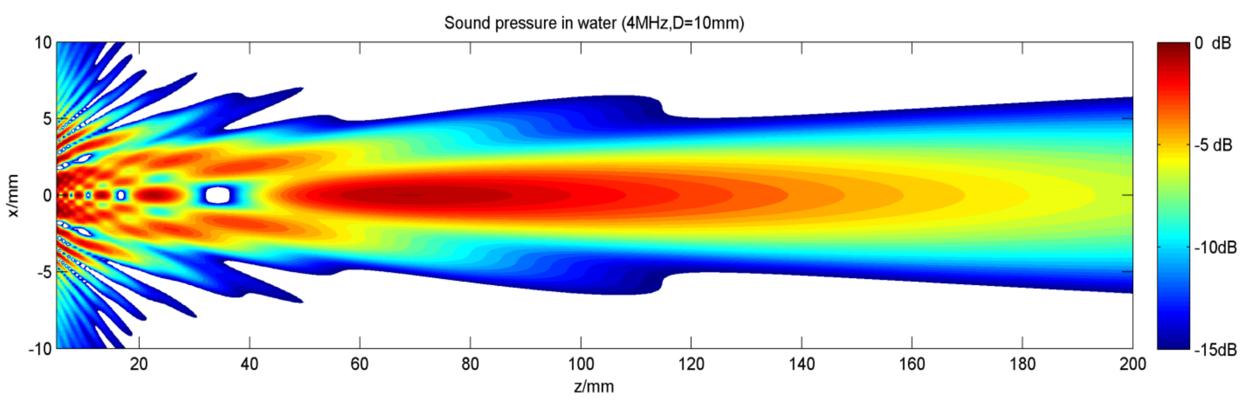



**Image credit:** Michael Lenz

# Parallel Linear Solver for Acoustic Problems with Rotationally Symmetric Boundary Surfaces

**Context: Vibrating structure immersed in fluid. Acoustic analysis using boundary element method.** Coupled to a finite element method for the structural analysis. We focus on the boundary element part of the calculation.

**Goal: Solve block circulant linear systems** to compute acoustic radiation of vibrating structure with **rotationally symmetric boundary surface.**

**Approach: Parallel linear solver** for **distributed memory machines** based on known inversion formula for block circulant matrices.

# THE BOUNDARY ELEMENT METHOD (BEM)

# Boundary Element Method

The **boundary element method (BEM)** is a numerical method for solving linear partial differential equations (PDEs).

In particular, the BEM is a solution method for **solving boundary value problems (BVPs) formulated using a boundary integral formulation.**

**Discretization:** Only of the surface (not of the volume). Reduces dimension of problem by one.

**BEM:** Used on **exterior domain problems** and when **greater accuracy** is required.

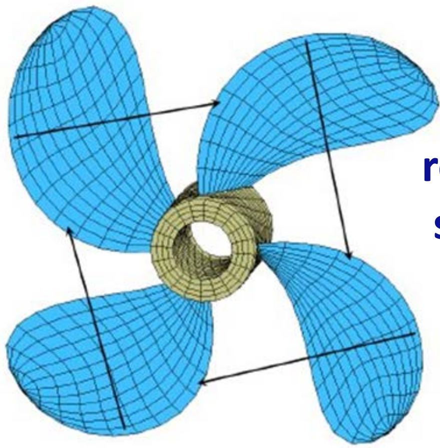
**We employ the boundary element method to obtain the linear system of equations.**

# Comparison of the BEM with the Finite Element Method

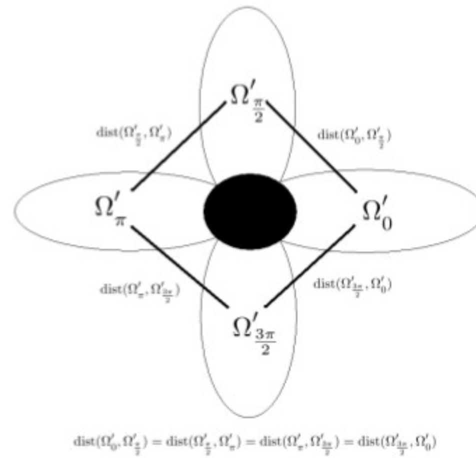
| Advantages of the BEM | Disadvantages of the BEM |
| --- | --- |
| **Less data preparation time** (due to surface only modeling) | **Unfamiliar mathematics** |
| **High resolution of PDE solution** (e.g., stress) | **The interior must be modeled for nonlinear problems** (but can often be restricted to a region of the domain) |
| **Less computer time and storage** (fewer nodes and elements) | **Fully populated and unsymmetric solution matrix** (as opposed to being sparse and symmetric) |
| **Less unwanted information** (most "interesting behavior" happens on the surface) | **Poor for thin structures (shell) 3D analyses** (large surface/volume ratio causes inaccuracies in calculations) |

# BLOCK CIRCULANT MATRICES VIA THE BOUNDARY ELEMENT METHOD

# Discretization Using the BEM



**rotationally symmetric boundary surface**




**symmetry: m = 4**

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_m \\ A_m & A_1 & \cdots & A_{m-1} \\ A_{m-1} & A_m & \cdots & A_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}$$

**block circulant matrix**

# Block Circulant Matrices

- **Properties of circulant matrices:** Diagonalizable by Fourier matrix. Can use DFT and IDFT. **Nice properties!**

- **Related work (serial):** algorithm derived from inversion formula (Vescovo, 1997); derivation (Smyrlis and Karageorghis, 2006)

- **Related work (parallel):** parallel block Toeplitz matrix solver (Alonso et al., 2005) (neglects potential concurrent calculations); parallel linear solver for axisymmetric case (Padiy and Neytcheva, 1997)

# MATHEMATICAL FORMULATION OF LINEAR SYSTEM OF EQUATIONS

# Notation: Fourier Matrix

The **Fourier matrix** is given by

$$F = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_m^1 & \omega_m^2 & \cdots & \omega_m^{m-1} \\ 1 & \omega_m^2 & \omega_m^4 & \cdots & \omega_m^{2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_m^{(m-1)} & \omega_m^{2(m-1)} & \cdots & \omega_m^{(m-1)^2} \end{bmatrix}$$

where $\omega_m = e^{i2\pi/m}$.

**Note:** The Fourier matrix is used in Fourier transforms.

# Discrete Fourier Transform (DFT)

**To compute the discrete Fourier transform (DFT) of a vector x, simply multiply F times x.**

**Example: m = 4**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

**F*x = u**

# Inverse Discrete Fourier Transform (IDFT)

**To compute the inverse of the discrete Fourier transform (IDFT) of a vector u, simply multiply $\frac{1}{m}$ F* times u, where F* = Hermitian of F.**

**Continuing the example:**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

**Divide by m**

**F* times u = m*x**

# Fast Fourier Transform (FFT)

The **Fourier transform** of a vector (i.e., the DFT of a vector) of length 2m **can be computed quickly** by taking advantage of the following **relationship between $F_m$ and $F_{2m}$:**

$$F_{2m} = \begin{pmatrix} I & D \\ I & -D \end{pmatrix} \begin{pmatrix} F_m & 0 \\ 0 & F_m \end{pmatrix} \mathrm{P},$$

where **D is a diagonal matrix** and **P is a 2m by 2m permutation matrix.**

**Fast Fourier Transform (FFT): Requires two size m Fourier transforms plus two very simple matrix multiplications!**

# Key Equations

Let F = Fourier matrix, and let $F_b$ denote the Kronecker product of F with $I_n$.

Then, the **block DFT is given by**:

$$\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \\ \vdots \\ \tilde{A}_m \end{bmatrix} = \begin{bmatrix} I_n & I_n & I_n & \cdots & I_n \\ I_n & I_n\omega_m^1 & I_n\omega_m^2 & \cdots & I_n\omega_m^{m-1} \\ I_n & I_n\omega_m^2 & I_n\omega_m^4 & \cdots & I_n\omega_m^{2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_n & I_n\omega_m^{(m-1)} & I_n\omega_m^{2(m-1)} & \cdots & I_n\omega_m^{(m-1)^2} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_m \end{bmatrix},$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$\tilde{X} \qquad\qquad F_b \qquad\qquad X$$

**To solve:** $\operatorname{diag}\{(\tilde{A}_1), (\tilde{A}_2), \ldots, (\tilde{A}_m)\}\tilde{x} = \tilde{b}$

where $\tilde{x} = F_b^* x, \tilde{b} = F_b^* b.$

# SERIAL ALGORITHM

# Block Circulant Matrix: Storage

**TABLE 1.** THE PERCENTAGE OF THE INITIAL COEFFICIENT MATRIX WHICH NEEDS TO BE STORED.

| $m$ | % of $A$ stored |
|---|---|
| 2 | 50% |
| 4 | 25% |
| 8 | 12.5% |
| 12 | 8.33% |
| 16 | 6.25% |

# Block Circulant Matrix: Size of Linear Systems

TABLE 2. SIZE OF THE LINEAR SYSTEMS FOR VARYING $m$ AND $N$.

| $N$ | $n, m = 4$ | $n, m = 8$ |
|---|---|---|
| 13,000 | 3,250 | 1,625 |
| 15,000 | 3,750 | 1,875 |
| 19,000 | 4,750 | 2,375 |
| 24,000 | 6,000 | 3,000 |

**Note: Solving a dense linear system is cubic in the size of the matrix.**

# Sequential Algorithm

**Algorithm 1** Pseudocode for the sequential solution of a block circulant linear system.

1. Compute $\tilde{b} = F_b^* b$. **IDFT**
2. Compute $\tilde{X} = F_b X$. **DFT**
3. Solve $\tilde{A}_j \tilde{x}_j = \tilde{b}_j$, $j = 1, \ldots, m$. **Solution of m independent linear systems**
4. Compute $x = F_b \tilde{x}/m$ **DFT**

# PARALLEL ALGORITHM

# How to Parallelize the Algorithm?

Ideas?

Recall, we are interested in solving

$$\text{diag}\{(\tilde{A}_1), (\tilde{A}_2), \ldots, (\tilde{A}_m)\}\tilde{x} = \tilde{b}.$$

**m independent linear systems to solve!**

where

$$\tilde{x} = F_b^* x, \tilde{b} = F_b^* b.$$

# Block DFT Algorithm

- A **block DFT calculation** is the basis for our **parallel algorithm**.
- This demonstrates **improved robustness** (over use of the FFT) and allows for **any boundary surface to be input**.

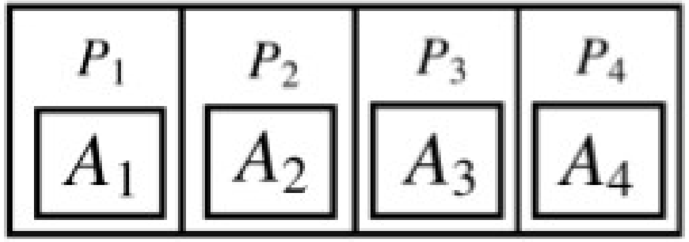



**FIGURE 4.** INITIAL DATA DISTRIBUTION ASSUMED IN THE DFT COMPUTATION FOR THE CASE $P = m = 4$.

# DFT Computation

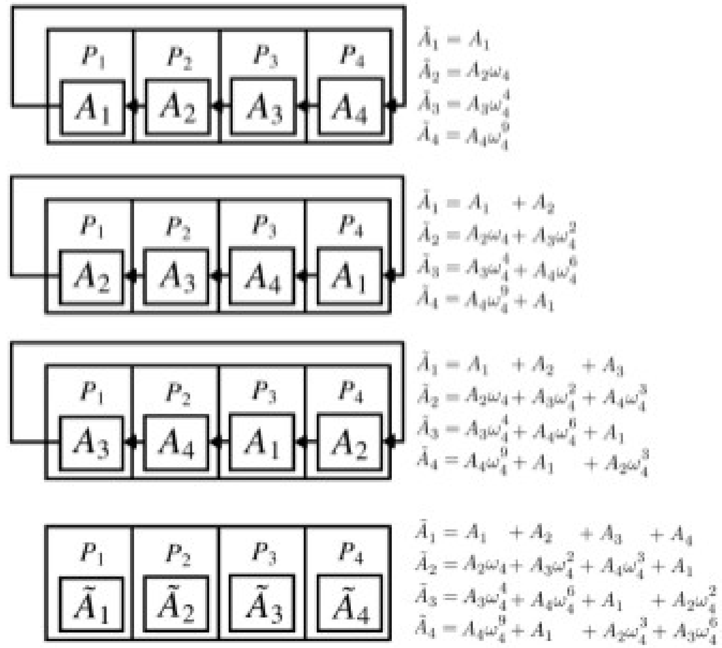



**FIGURE 5.** THE DFT COMPUTATION FOR THE CASE $P = m = 4$. EACH ARROW INDICATES THE COMMUNICATION OF A PROCESSOR'S OWNED SUBMATRIX TO A NEIGHBORING PROCESSOR IN THE DIRECTION OF THE ARROW.

**This generalizes to the case when P > m.**

# Parallel Algorithm

**Algorithm 2** Pseudocode for the parallel solution of a block circulant linear system, assuming $P = cm$.

1. Define $m$ $\sqrt{c} \times \sqrt{c}$ process grids.
2. Block cyclically distribute each $A_j$ and $b_j$ onto grid $G_j$ using identical blocking factors. **Asynchronous sends and receives**
3. Perform $c$ simultaneous IDFTs transforming $b_j$ to $\tilde{b}_j$.
4. Perform $c$ simultaneous DFTs transforming $A_j$ to $\tilde{A}_j$.
5. Simultaneously solve each $\tilde{A}_j \tilde{x}_j = \tilde{b}_j$ in parallel using PZGESV. **Solve using SCALAPACK. Complexity: Cubic in n**
6. Perform $c$ simultaneous DFTs transforming $\tilde{x}_j$ to $x_j$.

**The tradeoff of using overlapped communication and computation is additional memory.**

# NUMERICAL EXPERIMENTS

# Computer Architecture for Experiments

**Cyberstar compute cluster at Penn State:**

- Run on two Intel Xeon X5550 quad-core processors
- HyperThreading = disabled
- **Total:** 8 physical cores running at 2.66 GHz
- 24 GB of RAM per node

**Code:**

- Fortran 90 with MPI
- ScaLAPACK library

**Blocking and Communication:**

- Blocking factor of 50 for block cyclic distribution of $A_j$ and $b_j$ onto their respective processor grids.
- DFT algorithm communications: blocks of size 4000
- Asynchronous sends/receives

# Metrics for Experiments

- **Runtime** = wall clock time of parallel algorithm
  = $\mathbf{T_p}$

- **Speedup** = how much faster is the parallel algorithm than the serial algorithm = $\mathbf{S = T_s/T_p}$

- **Efficiency = E = S/P**

# Experimental Results – 4 Processors

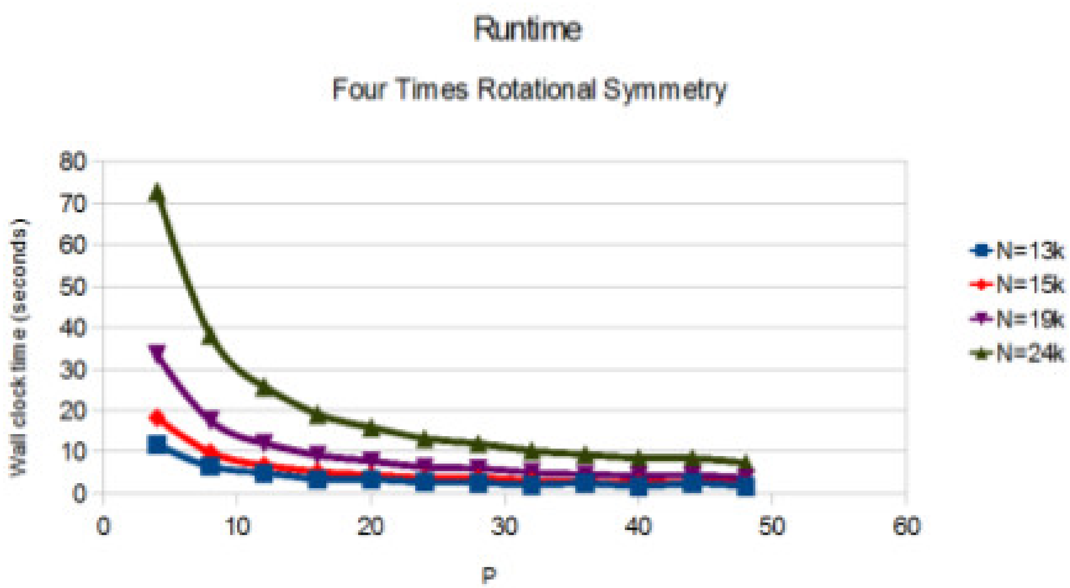



**FIGURE 9.** RUNTIME COMPARISON FOR VARYING $P$ AND $N$ WITH $m = 4$.

The runtime decreases as the number of processors increase and as the problem size decreases.

# Experimental Results – 4 Processors

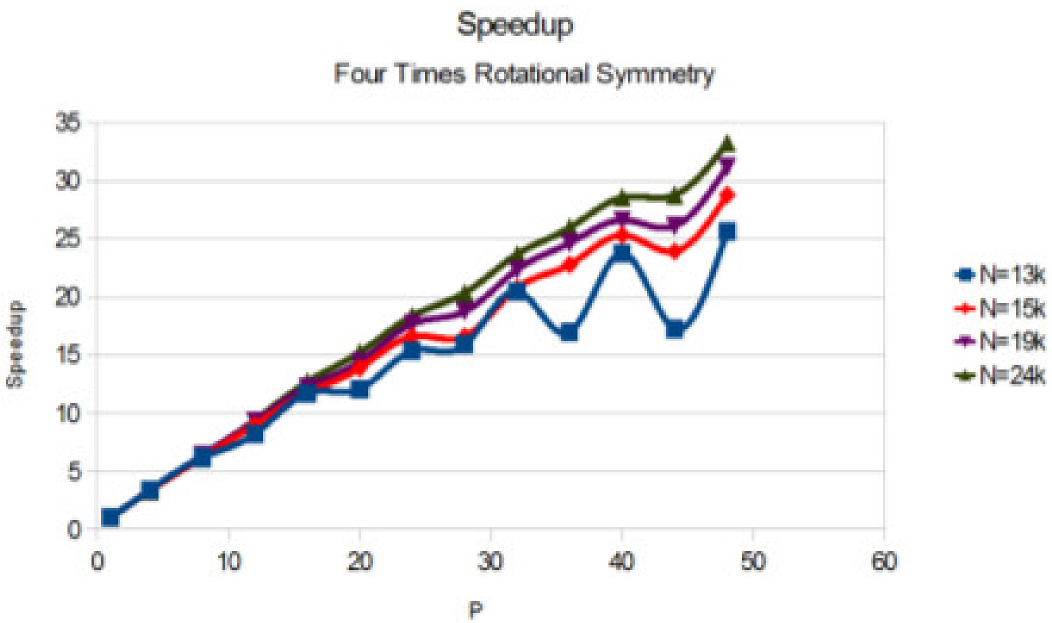



**FIGURE 10.** SPEEDUP COMPARISON FOR VARYING $P$ AND $N$ WITH $m = 4$.

**Oscillations are due to small variance in small runtime numbers. They are smoothed out with increasing N.**

# Experimental Results – 4 Processors

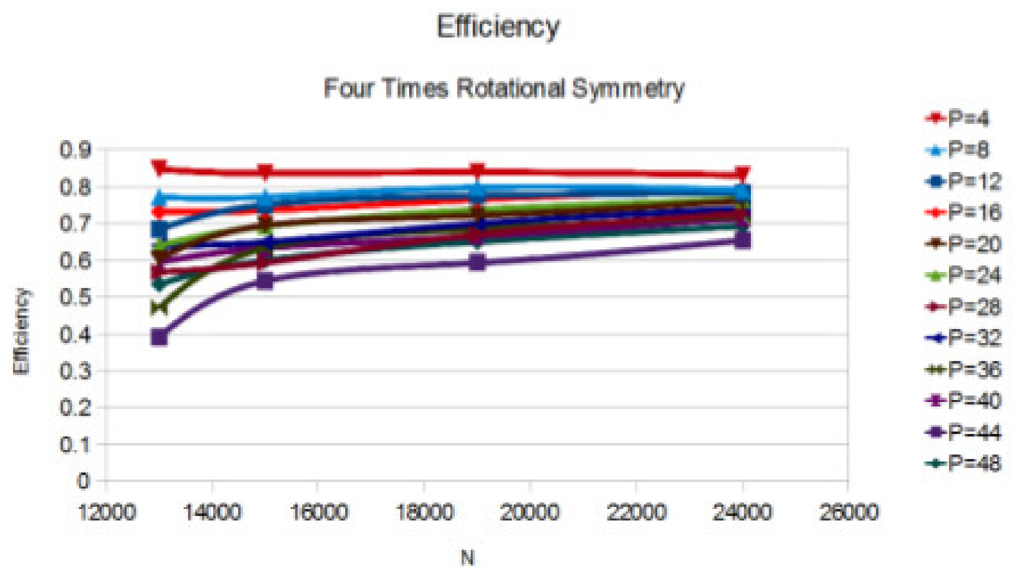



**FIGURE 11.** EFFICIENCY COMPARISON FOR VARYING $N$ AND $P$ WITH $m = 4$.

**The efficiency increases for a decreased number of processors. It also increases with an increase in problem size.**

# Experimental Results – 8 Processors

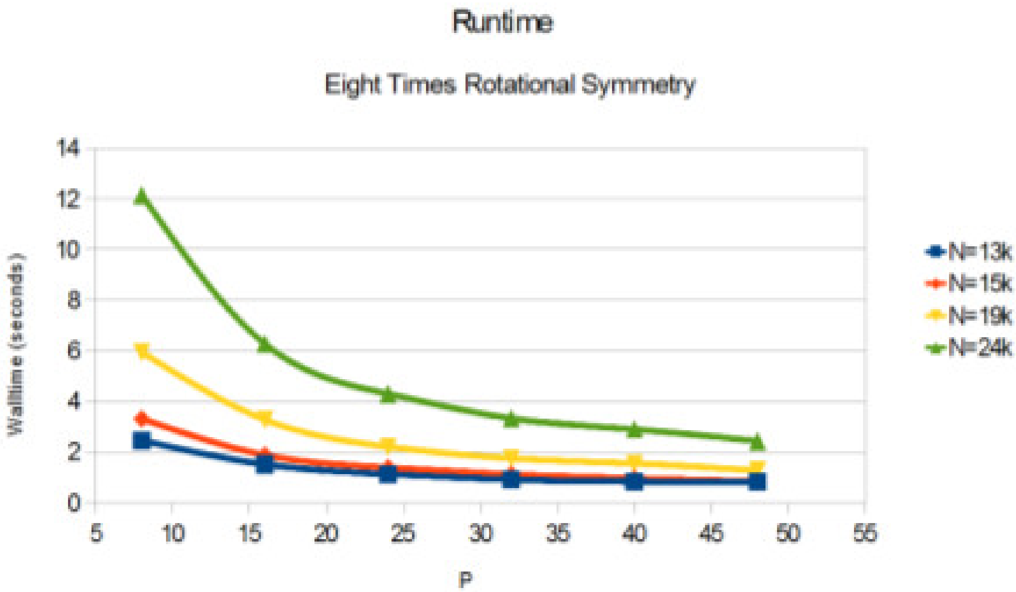



**FIGURE 12.** RUNTIME COMPARISON FOR VARYING $P$ AND $N$ WITH $m = 8$.

**The runtime trend is the same as it is for m = 4.**

# Experimental Results – 8 Processors

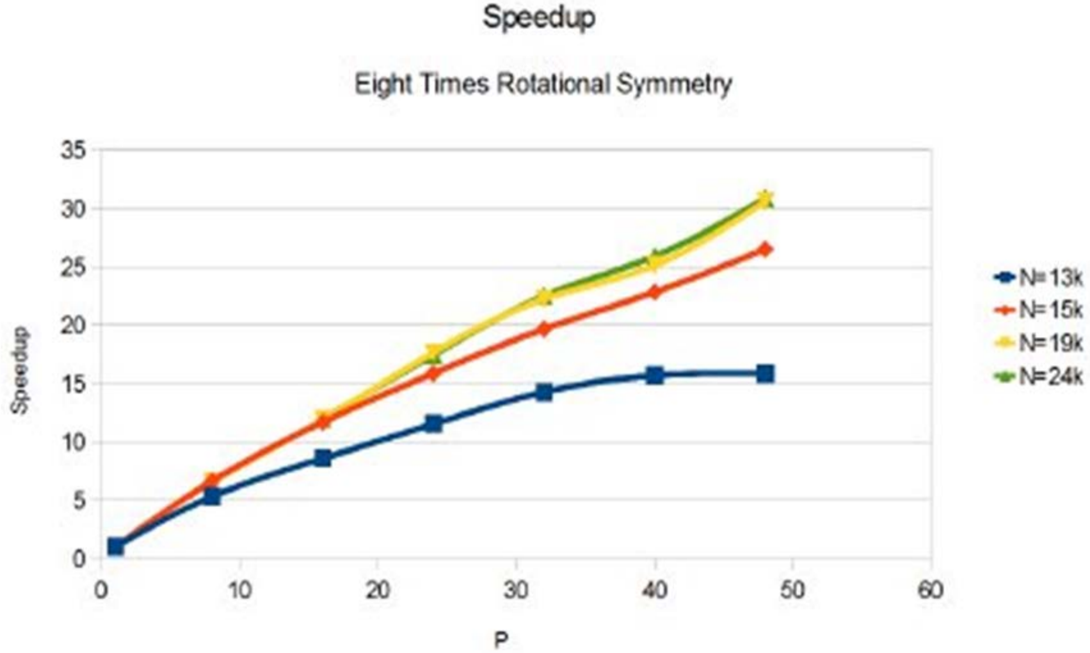



**FIGURE 13.** SPEEDUP COMPARISON FOR VARYING $P$ AND $N$ WITH $m = 8$.

**For small problems, the speedup levels off due to the ratio of computation versus communication in the linear system solve. For larger problems, the speedup is nearly linear.**

# Experimental Results – 8 Processors

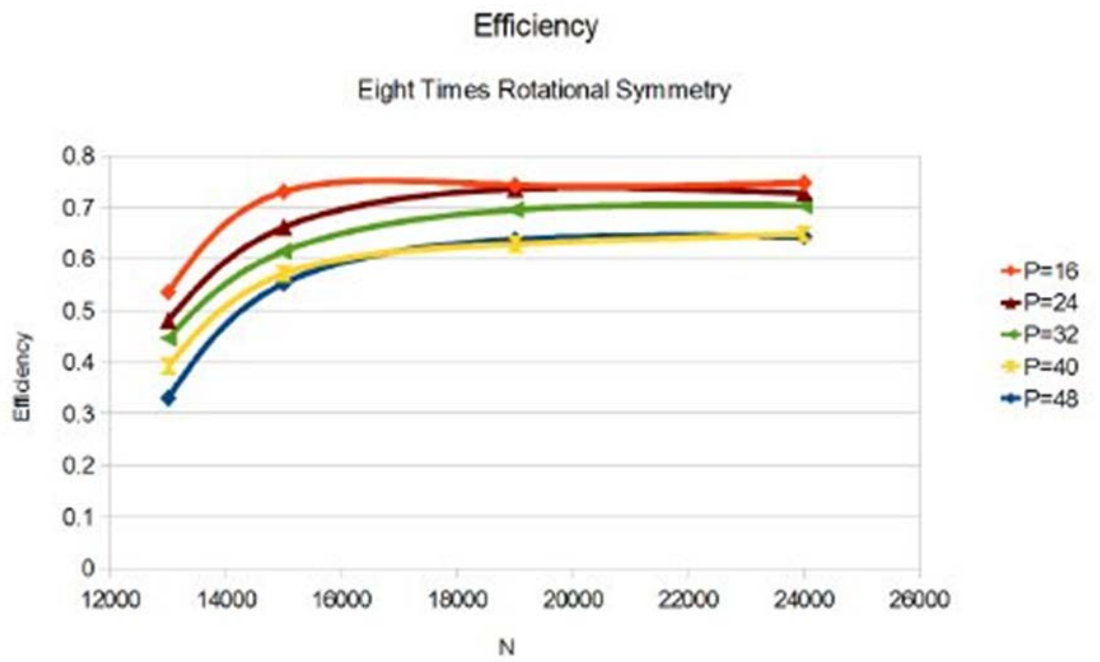



**FIGURE 14.** EFFICIENCY COMPARISON FOR VARYING $P$ AND $N$ WITH $m = 8$.

**The efficiency is fairly good but not quite as high as it is for m = 4. Based on increase in communications due to DFT algorithm and size of linear system solve. Expect efficiency to remain high for increased problem size.**

# CONCLUSIONS

# Conclusions

- We have proposed a **parallel algorithm** for **solution of block circulant linear systems**. Arise from **acoustic radiation problems** with **rotationally symmetric boundary surfaces**.
- Based on **block DFTs (more robust)** and have **embarrassingly parallel nature** based on ScaLAPACK's required data distributions.
- **Reduced memory requirement** by **exploiting block circulant structure.**
- Achieved near linear speedup for varying problem size, **linear speedup for large N. Efficiency increases with problem size.**
- **Can solve larger/higher frequency** acoustic radiation **problems.**

# Reference

K.D. Czuprynski, J. Fahnline, and S.M. Shontz, *Parallel boundary element solutions of block circulant linear systems for acoustic radiation problems with rotationally symmetric boundary surfaces*, Proc. of the Internoise 2012/ASME NCAD Meeting, August 2012.

# Acknowledgements

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