EECS 801: Midterm Exam

Due: Tuesday, February 23, 2016 (At the beginning of lecture)

Exam rules: This exam has four questions and is two pages long. The questions are weighted equally. The exam counts for 15% of your final course grade. The exam is due at 1pm on Tuesday, February 23. You must turn-in your exam solutions in person at the beginning of lecture. Exams found under my door, in my mailbox, etc., will not be accepted.

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than the textbooks, you must cite them.

Academic integrity: You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until 1pm on Tuesday, February 23. Please write and sign and date the following statement on a cover page to be submitted with your exam solutions: "I have neither given nor received unpermitted assistance on this exam."

You are not allowed to send any e-mail or otherwise make any on-line posting concerning the questions on this exam until after it is over. But you are allowed to consult publicly-available websites and search engines.

Help from the instructor: The only help available will be clarification of the questions. If you have questions, I can be reached via e-mail at shontz@ku.edu or via cell at 785-764-6980. No help will be given towards finding a solution.

Late policy: No late exam solutions will be accepted. The solutions must be turned in by 1pm on Tuesday, February 23.

Questions:

1. (20 points) Consider the upwind method $u_j^{n+1} = u_j^n + \lambda\left(u_{j+1}^n - u_j^n\right)$, where $\lambda = \Delta t/\Delta x$, for the first-order wave equation, $u_t = u_x$, which is a first-order method. Show that this method is actually second-order for a different PDE, namely, for a certain advection-diffusion equation $u_t = u_x + au_{xx}$, where a may depend on Δt and/or Δx.

2. (20 points) For the linear advection equation $u_t + au_x = 0$, where a is a positive constant, the Lax-Friedrichs scheme on a uniform mesh is defined by

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\alpha}{2}\left(u_{j+1}^n - u_{j-1}^n\right),$$

   where $\alpha = a\Delta t/\Delta x$.

   Analyze the CFL condition to determine if there are any restrictions on Δt for stability. Similarly, does the von Neumann stability analysis indicate any restrictions on Δt are needed for stability? What is the truncation error of this scheme, and how does it behave as Δt increases?

3. (20 points) Consider the boundary value problem defined by the following system of equations:

$$\begin{aligned} -\nabla^2 u + u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma, \end{aligned}$$

where Ω is a domain with boundary Γ.

Show that the corresponding variational and minimization problems for the solution of this boundary value problem are equivalent.

4. (20 points) Let Ω be a polygonal domain in the plane. Consider the system of equations

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here λ is an unknown scalar, and u is an unknown scalar-valued function on Ω. The problem is to find nontrivial solutions (λ, u) to this problem. ("Nontrivial" means solutions where u is not identically zero.) This problem is called an "eigenvalue problem". Show how to discretize it by applying various finite element techniques from lecture such as Green's Theorem. Your goal is to transform the PDE to a discrete linear algebra problem of the following form: Find solutions to $K\mathbf{u} = \lambda M\mathbf{u}$, where K and M are given $n \times n$ matrices, λ is an unknown scalar, and $\mathbf{u}$ is an unknown n-vector. It is not necessary to propose an algorithm for solving the discrete problem.

Verify that in your discretized problem, the matrix K you derive is the same as the assembled stiffness matrix for the finite element method for Poisson's equation with Dirichlet boundary conditions. Also, show that the matrix M you end up with is symmetric, positive definite and sparse.