EECS 801: Homework 1

Due: Thursday, February 4, 2016 (At the beginning of lecture)

Reminder: Please review the homework policies on the syllabus before working on the assignment.

Questions:

1. (20 points) Consider the three-dimensional Poisson's problem

$$
\begin{aligned}
-(u_{xx} + u_{yy} + u_{zz}) = f(x, y, z) \ \in \Omega \quad &= \quad (0,1)^3, \\
u|_{\partial\Omega} &= 0.
\end{aligned}
$$

   Derive the seven-point scheme for the above PDE with uniform spacing in a particular direction. Use h, k, and p for the spacing with $h \neq k \neq p$.

2. (20 points) Develop a second-order difference scheme to solve the biharmonic problem

$$
\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0, \ (x, y) \in \Omega = (0,1)^2,
$$

$$
u = f(x, y), \ \frac{\partial u}{\partial n} = g(x, y), \text{on } \partial\Omega.
$$

   Implement your scheme and test it with proper f and g such that the exact solution $u = x^3 y^3$. What do you notice about your numerical solution as you refine your mesh?

3. (20 points) Sketch the characteristics for the equation $u_t + au_x = 0$ for $0 \leq x \leq 1$ when $a = a(x) = x - 1/2$. Set up the upwind scheme on a uniform mesh $\{x_j = j\Delta x, j = 0, 1, \ldots, J\}$, noting that no boundary conditions are needed, and derive an error bound; consider both even and odd J. Sketch the development of the solution when $u(x, 0) = x(1 - x)$ and obtain explicit error bounds by estimating the terms in the trunctation error.

   Repeat the exercise with $a(x) = 1/2 - x$, but with boundary conditions $u(0, t) = u(1, t) = 0$.

4. (20 points) For the linear advection equation $u_t + au_x = 0$, where a is a positive constant, a generalised upwind scheme on a uniform mesh is defined by

$$
U_j^{n+1} = (1 - \theta)U_k^n + \theta U_{k-1}^n,
$$

   where $x_k - \theta\Delta x = x_j - a\Delta t$ and $0 \leq \theta < 1$. Very that the CFL condition requires no restriction on Δt, and that the von Neumann stability analysis also shows that stability is unrestricted. What is the truncation error of this scheme, and how does it behave as Δt increases?