Projection Matrix Summary

0. Recall, we build all 4x4 matrices here as a product: $\mathbf{M}_{ec\text{-}lds} = \mathbf{M}_{wv} * \mathbf{M}_{proj}$ where:

   $\mathbf{M}_{wv}$ does the window-viewport map into the -1..+1 logical device space of OpenGL

   $\mathbf{M}_{proj}$ does the 3D to 2D projection with preservation of (at least relative) depth.

**1.** **Orthogonal** (Given: $x_{min}$, $x_{max}$, $y_{min}$, $y_{max}$, $z_{min}$, $z_{max}$, all specified in eye coordinates with $x_{min}<x_{max}$; $y_{min}<y_{max}$; and $z_{min}<z_{max}$)

   $\mathbf{M}_{proj}$ is the identity matrix since there is nothing that needs to be done. $\mathbf{M}_{wv}$ simply maps ($x_{min}$, $x_{max}$, $y_{min}$, $y_{max}$, $z_{min}$, $z_{max}$) to (-1, 1, -1, 1, 1, -1). (Note the reversal in the $z$ direction.)

   This yields three pairs of equations with two unknowns:

$$a_x x_{min} + b_x = -1 \text{ and } a_x x_{max} + b_x = 1$$
$$a_y y_{min} + b_y = -1 \text{ and } a_y y_{max} + b_y = 1$$
$$a_z z_{min} + b_z = 1 \text{ and } a_z z_{max} + b_z = -1$$

   Solving for $a_x$, $b_x$, $a_y$, $b_y$, $a_z$, and $b_z$, we get:

$$\begin{aligned} a_x &= 2/(x_{\max} - x_{\min});\ b_x = -(x_{\max} + x_{\min})/(x_{\max} - x_{\min}) \\ a_y &= 2/(y_{\max} - y_{\min});\ b_y = -(y_{\max} + y_{\min})/(y_{\max} - y_{\min}) \\ a_z &= -2/(z_{\max} - z_{\min});\ b_z = (z_{\max} + z_{\min})/(z_{\max} - z_{\min}) \end{aligned} \qquad (1)$$

   Hence

$$\mathbf{M}_{wv} = \begin{pmatrix} a_x & 0 & 0 & b_x \\ 0 & a_y & 0 & b_y \\ 0 & 0 & a_z & b_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   Finally, $\mathbf{M}_{ec\text{-}lds} = \mathbf{M}_{wv}\mathbf{M}_{proj} = \mathbf{M}_{wv}\mathbf{I} = \mathbf{M}_{wv}$.

**2.** **Oblique** (Given: $z_{pp}$, $x_{min}$, $x_{max}$, $y_{min}$, $y_{max}$, $z_{min}$, $z_{max}$, and $\mathbf{d}=(d_x, d_y, d_z)$, the common direction of projection, all specified in eye coordinates with $x_{min}<x_{max}$; $y_{min}<y_{max}$; $z_{min}<z_{max}$; and $d_z \neq 0$)

   $\mathbf{M}_{proj}$ can be shown to be:

$$\mathbf{M}_{proj} = \begin{pmatrix} 1 & 0 & -d_x/d_z & z_{pp}d_x/d_z \\ 0 & 1 & -d_y/d_z & z_{pp}d_y/d_z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   Clearly $\mathbf{M}_{wv}$ is the same for oblique as for orthogonal, hence:

$$\mathbf{M}_{ec-lds} = \mathbf{M}_{wv}\mathbf{M}_{proj} = \begin{pmatrix} a_x & 0 & \frac{-a_x d_x}{d_z} & \frac{a_x z_{pp} d_x}{d_z} + b_x \\ 0 & a_y & \frac{-a_y d_y}{d_z} & \frac{a_y z_{pp} d_y}{d_z} + b_y \\ 0 & 0 & a_z & b_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a_x$, $b_x$, $a_y$, $b_y$, $a_z$, and $b_z$ are as given in equation (1) above.

3. **Perspective** (Given: $z_{pp}$, $x_{min}$, $x_{max}$, $y_{min}$, $y_{max}$, $z_{min}$, $z_{max}$, all specified in eye coordinates with $x_{min}<x_{max}$; $y_{min}<y_{max}$; $z_{min}<z_{max}<0$; and $z_{pp}<0$)

We derive $\mathbf{M}_{proj}$ (and, in particular, the portions of the transformation involving the eye coordinate $z$ direction) so that mapping to the $z$ range of LDS space is included in $\mathbf{M}_{proj}$. Thus we get:

$$\mathbf{M}_{wv} = \begin{pmatrix} a_x & 0 & 0 & b_x \\ 0 & a_y & 0 & b_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{M}_{proj} = \begin{pmatrix} z_{pp} & 0 & 0 & 0 \\ 0 & z_{pp} & 0 & 0 \\ 0 & 0 & \alpha_z & \beta_z \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $a_x$, $b_x$, $a_y$, and $b_y$ are as given in equation (1) above. The $\alpha_z$ and $\beta_z$ terms can be shown to be:

$$\alpha_z = -\frac{z_{min} + z_{max}}{z_{max} - z_{min}}; \; \beta_z = \frac{2z_{min} z_{max}}{z_{max} - z_{min}}$$

Finally:

$$\mathbf{M}_{ec-lds} = \mathbf{M}_{wv}\mathbf{M}_{proj} = \begin{pmatrix} a_x & 0 & 0 & b_x \\ 0 & a_y & 0 & b_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_{pp} & 0 & 0 & 0 \\ 0 & z_{pp} & 0 & 0 \\ 0 & 0 & \alpha_z & \beta_z \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_x z_{pp} & 0 & b_x & 0 \\ 0 & a_y z_{pp} & b_y & 0 \\ 0 & 0 & \alpha_z & \beta_z \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In principle, this matrix should be fine, but there is a clipping issue we will discuss that forces us to use the negated version of this matrix. Basically we need to be sure that the $w$ component that results when this matrix is used is positive for any points in the view frustum. Since this matrix will set $w=z$, all visible points will have negative $w$. Negating the matrix prevents that without altering how points are projected since negating all 16 elements will just produce a different (but *projectively equivalent*) point. Hence:

$$\mathbf{M}_{ec-lds} = \begin{pmatrix} -a_x z_{pp} & 0 & -b_x & 0 \\ 0 & -a_y z_{pp} & -b_y & 0 \\ 0 & 0 & -\alpha_z & -\beta_z \\ 0 & 0 & -1 & 0 \end{pmatrix}$$