Operations on Sets:

Read: Chpt. 2.2

Simple Set Operations:
Given (simple) sets A, B, C, ….

Def. The ***union*** of A and B:
$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$.

Def. The ***intersection*** of A and B:
$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$.

Def. Two sets A and B are ***disjoint*** iff $A \cap B = \emptyset$.

A Simple Counting Problem:
How are $|A|$, $|B|$, $|A \cup B|$, $|A \cap B|$ related?

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is the simplest form of the ***Principle of Inclusion–Exclusion***.

Venn Diagrams:

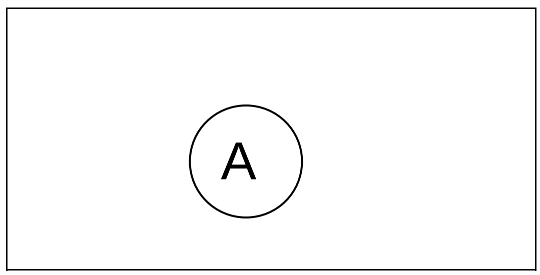
A graphical representation of sets with:

Universal set $\leftrightarrow$ Rectangle

Sets $\leftrightarrow$ Geometric objects inside the rectangle

Set Relation $\leftrightarrow$ Intersection of geometric objects

Example: A Venn diagram representing a simple set A.



U: universal set

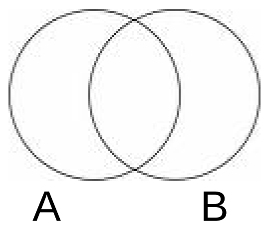
Remark: When exploring some general relations among several sets using Venn diagram, all the geometric objects/sets must intersect each other and none of them should be wholly contained in any one of them in the Venn diagram.

HW. Review Venn diagrams in book.

Recall that from the Principle of Inclusion–Exclusion, we have

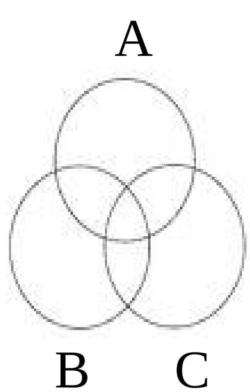
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Consider the following Venn diagram for A and B.



Q: Can you see why the above identity holds?

Consider the following Venn diagram for A, B, and C.



Q: Can you see an extension of the above identity?

Warning: Venn diagram is merely a graphical tool used in illustration only. You can not prove any set identity using Venn diagram!!!

Def. The ***difference*** of A and B:

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}.$$

Observe that, in general,

$$A - B \neq B - A.$$

Def. The ***symmetric difference*** of A and B:

$A \oplus B$ = a set of elements x with $x \in A$ <u>or</u> $x \in B$

$= \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$

$= (A - B) \cup (B - A)$

Def. If the universal set U is specified, we can define the ***complement*** of A to be

$$\overline{A} = U - A = \{x \mid (x \in U) \wedge (x \notin A)\}.$$

Example:

Given U = {b, 2, 1, c, a, 6, 7, 8}, A = {a, 2, 8}, B = {1, 2, 8, b}, D = {a, b, c}.

$A - B = \{a\}$,

$B - A = \{1, b\}$,

$A \oplus B = \{a, 1, b\}$,

$\overline{D} = \{1, 2, 6, 7, 8\}$,

$A \cup B = \{1, 2, 8, a, b\}$, $|A \cup B| = 5$,

$A \cap B = \{2, 8\}$, $|A \cap B| = 2$,

$|A \cup B| = |A| + |B| - |A \cap B| = 3 + 4 - 2 = 5.$

Q: How do we prove the equality of two set expressions?

Proving Set Identities:

Some Possible Approaches:

1. Using set definitions and direct proof technique.
2. Using Laws of Logical Equivalence for propositions.
3. Using truth tables.
4. Using set identities.

Examples:

1. Using the definition of set (operations), prove that $A \cap (A \cup B) = A$.

   Proof. We will prove that (i) $A \cap (A \cup B) \subseteq A$, and (ii) $A \subseteq A \cap (A \cup B)$.

   (i) Let $x \in A \cap (A \cup B)$. By definition of sets intersection, $x \in A$ and $x \in A \cup B$. Hence, $A \cap (A \cup B) \subseteq A$.

   (ii) Let $x \in A$. If $x \notin A \cap (A \cup B)$, then $x \notin A$ or $x \notin A \cup B$. By assumption, $x \in A$, implying that $x \in A$ and $x \in A \cup B$. Hence, a contradiction is reached and $x \in A \cap (A \cup B)$.

   Since $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$, we have proved that $A \cap (A \cup B) = A$.

2. Prove that $A \cap (A \cup B) = A$ using Laws of Equivalence for propositions.

Proof.

$A \cap (A \cup B)$	
$= \{x \mid x \in A \cap (A \cup B)\}$	**Def of $A \cap (A \cup B)$**
$= \{x \mid (x \in A) \wedge (x \in (A \cup B))\}$	**Def of $\cap$**
$= \{x \mid (x \in A) \wedge ((x \in A) \vee (x \in B))\}$	**Def of $\cup$**
$= \{x \mid x \in A\}$	**Absorption Law**
$= A$	

3. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.

Proof.

$\overline{(A \cup B)}$	
$= \{x \mid \neg(x \in A \vee x \in B)\}$	**Def. of $\overline{(A \cup B)}$**
$= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$	**De Morgan(log. eq.)**
$= \{x \mid x \notin A \wedge x \notin B\}$	**Def. of negation**
$= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\}$	**Def. of set complement**
$= \overline{A} \cap \overline{B}$.	**Def. of set intersection**

4. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$A \cap (B \cup C)$

$= \{x \mid (x \in A) \wedge x \in (B \cup C)\}$ **Def. of $A \cap (B \cup C)$**

$= \{x \mid (x \in A) \wedge ((x \in B) \vee (x \in C))$

Def. of $B \cup C$

$= \{x \mid ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))\}$

Distrib. Law (log. eq.)

$= \{x \mid (x \in A \cap B) \vee (x \in A \cap C)\}$

Def. of set intersection

$= (A \cap B) \cup (A \cap C)$ **Def. of set union**

Observe that above proofs are based on the information of whether an arbitrarily given element belongs to a set. Hence, we can also prove these identities using a membership table, which is similar to a truth table but with the following modifications.

Truth Table	**Membership Table**
Proposition	Set
T	1
F	0

5. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.

A	**B**	$\overline{A}$	$\overline{B}$	**A ∪ B**	$\overline{(A \cup B)}$	$\overline{A} \cap \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

6. A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

A	**B**	**C**	**B∪C**	**A∩B**	**A∩C**	**A∩(B∪C)**	**(A∩B)∪(A∩C)**
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Some Useful Set Identities:

1. $A \cup \varnothing = A$ — **Identity Laws**
   $A \cap U = A$

2. $A \cup U = U$ — **Domination Laws**
   $A \cap \varnothing = \varnothing$

3. $A \cup A = A$ — **Idempotent Laws**
   $A \cap A = A$

4. $(\overline{\overline{A}}) = A.$ — **Involution Law**

5. $A \cup \overline{A} = U$ — **Complement Laws**
   $A \cap \overline{A} = \varnothing$

6. $A \cup (B \cup C) = (A \cup B) \cup C$ — **Associative Laws**
   $A \cap (B \cap C) = (A \cap B) \cap C$

7. $A \cup B = B \cup A$ — **Commutative Laws**
   $A \cap B = B \cap A$

8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ — **Distributive Laws**
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

9. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ — **De Morgan's Law**
   $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Remark:

The set identities above can be obtained from the corresponding laws of logical equivalence by the following transformation.

Logical Equivalence	**Set Identity**
Proposition	Set
$\wedge$	$\cap$
$\vee$	$\cup$
T	U
F	$\varnothing$
Negation	Complement

Proving Set Equality Using Set Identities:

1. $A \cup \overline{(A \cap B)} = U$

$A \cup \overline{(A \cap B)}$

$= A \cup (\overline{A} \cup \overline{B})$ **De Morgan's Law**

$= (A \cup \overline{A}) \cup \overline{B}$ **Associative Law**

$= U \cup \overline{B}$ **Complement Law**

$= U$ **Domination Law**

2. $(\overline{A} \cup B) \cup \overline{(A \cap B)} = U$

$(\overline{A} \cup B) \cup \overline{(A \cap B)}$

$= (\overline{A} \cup B) \cup (\overline{A} \cup \overline{B})$ **De Morgan's Law**

$= (\overline{A} \cup \overline{A}) \cup (B \cup \overline{B})$ **Associative Law**

$= \overline{A} \cup U$ **Complement Law**

$= U$ **Domination Law**

Generalized Unions and Intersections of Sets:

The union and intersection operations can be extended to a finite number of sets.

Def. The ***union*** of a collection of sets $A_1, A_2, \ldots, A_n$:

$A_1 \cup A_2 \cup \ldots \cup A_n$

$= \{x \mid (x \in A_1) \vee (x \in A_2) \vee \ldots \vee (x \in A_n)\}$

$= \bigcup_{i=1}^{n} A_i$.

Def. The ***intersection*** of a collection of sets $A_1, A_2, \ldots, A_n$:

$A_1 \cap A_2 \cap \ldots \cap A_n$

$= \{x \mid (x \in A_1) \wedge (x \in A_2) \wedge \ldots \wedge (x \in A_n)\}$

$= \bigcap_{i=1}^{n} A_i$.

Remark: The associated and commutative laws can also be extended to any finite number of sets.

10. **Generalized De Morgan's Law**

$$\overline{(A \cap B \cap C \cap \ldots)} = \overline{A} \cup \overline{B} \cup \overline{C} \cup \ldots$$

$$\overline{(A \cup B \cup C \cup \ldots)} = \overline{A} \cap \overline{B} \cap \overline{C} \cap \ldots$$

Practice HW: Chpt.2.2: 3, 7, 9, 13, 17, 19, 25, 27, 29, 31, 37, 39.