(a) Determine the system function $H(z)$ for the causal LTI system with difference equation

$$y[n] - \frac{1}{2}y[n-1] + \frac{1}{4}y[n-2] = x[n]$$

(b) Using z-transforms, determine $y[n]$ if

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Taking the z-transform of the diffeence equation we get

$$Y(z) - \frac{1}{2}z^{-1}Y(z) + \frac{1}{4}z^{-2}Y(z) = X(z)$$

$$\Rightarrow H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} = \frac{z^2}{\underbrace{z^2-\frac{1}{2}z+\frac{1}{4}}_{\text{roots} = \frac{1}{4} \pm \frac{\sqrt{3}}{4}}}$$

$$= \frac{z^2}{\left(z-\left(\frac{1}{4}+j\frac{\sqrt{3}}{4}\right)\right)\left(z-\left(\frac{1}{4}-j\frac{\sqrt{3}}{4}\right)\right)}$$

Taking the z-transform of $x[n]$ we get

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$Y(z) = H(z)X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \times \frac{1}{1-\frac{1}{2}z+\frac{1}{4}z^{-2}}$$

$$= \frac{A}{\underbrace{1-\frac{1}{2}z^{-1}}_{\text{O+W Table 10.2 pair\# 5}}} + \frac{B}{\underbrace{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}_{\text{O+W Table 10.2 pair \# 12}}}$$

$A = 1$

$B = z^{-1}/2$

$\Rightarrow r = 1/2$

$\omega_0 = \pi/3$

$$\boxed{y[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^n \sin(\pi/3\, n)\, u[n]}$$

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