We have shown that if a system is LTI then the output can be described by convolution. We will now show that if a system can be described by convolution then it is LTI

Notation

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$x(t) \longrightarrow y(t)$$

$\boxed{\text{can be repeated for discrete time}}$

Let $x_1(t) = g(t) \longrightarrow y_1(t) = \int_{-\infty}^{\infty} g(t-\tau)\,h(\tau)\,d\tau$

Let $x_2(t) = w(t) \longrightarrow y_2(t) = \int_{-\infty}^{\infty} w(t-\tau)\,h(\tau)\,d\tau$

Let $x_3(t) = \alpha x_1(t) + \beta x_2(t)$

$$\longrightarrow y_3(t) = \int_{-\infty}^{\infty} \big(\alpha g(t-\tau) + \beta w(t-\tau)\big)\,h(\tau)\,d\tau$$

$$\boxed{= \alpha y_1(t) + \beta y_2(t)}$$

The system is additive and homogeneous and is therefore $\boxed{\text{Linear}}$

Notation

$$y[n] = \sum_{m=-\infty}^{\infty} x[n-m]\,h[m]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$x[n] \longrightarrow y[n]$$

$\boxed{\text{Can be repeated for continuous time}}$

Let $x_1[n] = g[n] \rightarrow y_1[n] = \sum_{m=-\infty}^{\infty} g[n-m]\,h[m]$

Let $x_2[n] = g[n-n_0] = x_1[n-n_0]$

$$\rightarrow y_2[n] = \sum_{m=-\infty}^{\infty} g[n-n_0-m]\,h[m]$$

$$\boxed{= y_1[n-n_0]}$$

The system is $\boxed{\text{time invariant}}$