Approximation of the Continuous Time Fourier Transform

Signals & Systems

Lab 8

Continuous Time Fourier Transform (CTFT)

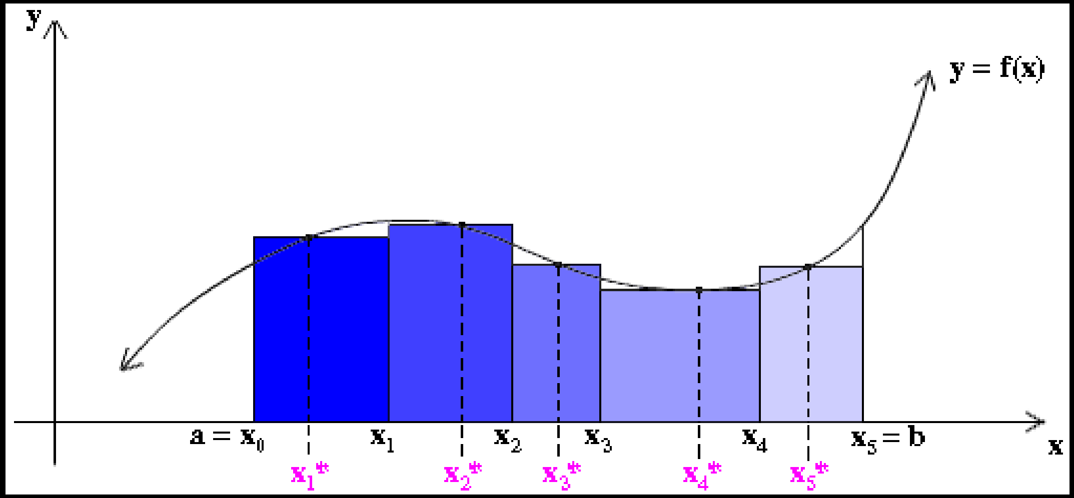
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

“Uncle” Fourier

Riemann Definite Integral

$$\int_a^b f(x)dx \triangleq \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$



“Uncle” Riemann

Riemann Definite Integral

The limit $\Delta x \to 0$ means that the total interval $[a, b]$ will be filled with a infinite number of subintervals each with width Δx!

$$\int_a^b f(x)dx \triangleq \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\, \Delta x_i$$

$f(x)$ is defined where x $\in [a, b]$

$\Delta x_i = \Delta x = \frac{b-a}{n}$ called the subinterval (assumed all equal in our case)

n is the number of subintervals which “fill” $[a, b]$

x_i^* are called the sample points of f for each subinterval

As MATLAB can realistically operate only on discrete data we would like to use this definition to find an approximation to the CTFT.

In light of the previous observation we would like to express the Fourier Transform integral as a sum,

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt \triangleq \lim_{\Delta t \to 0} \sum_{i=1}^{n} x(t_i^*)e^{-j2\pi f t_i^*}\Delta t$$

But in this form the expression for the Fourier transform is still impractical because it requires an <u>infinite</u> number of terms.

Time-Domain Housekeeping

Now $x(t)$ is a signal in continuous time. But in MATLAB it will take on amplitude values only at discrete time points which depend on when the continuous-time signal was sampled.

So suppose $x(t)$ is being sampled every Δt seconds resulting in a sampled function $x[m] = x(m\Delta t)$ where our time index, m, start at 0 (causal) and ends at N, the total number of samples.

Time-Domain Housekeeping

Hence the Fourier transform of this $x[m]$ signal is

$$X(f) = \lim_{\Delta t \to 0} \sum_{m=0}^{N-1} x(m\Delta t) e^{-j2\pi f m\Delta t} \Delta t$$

Time-Domain Housekeeping

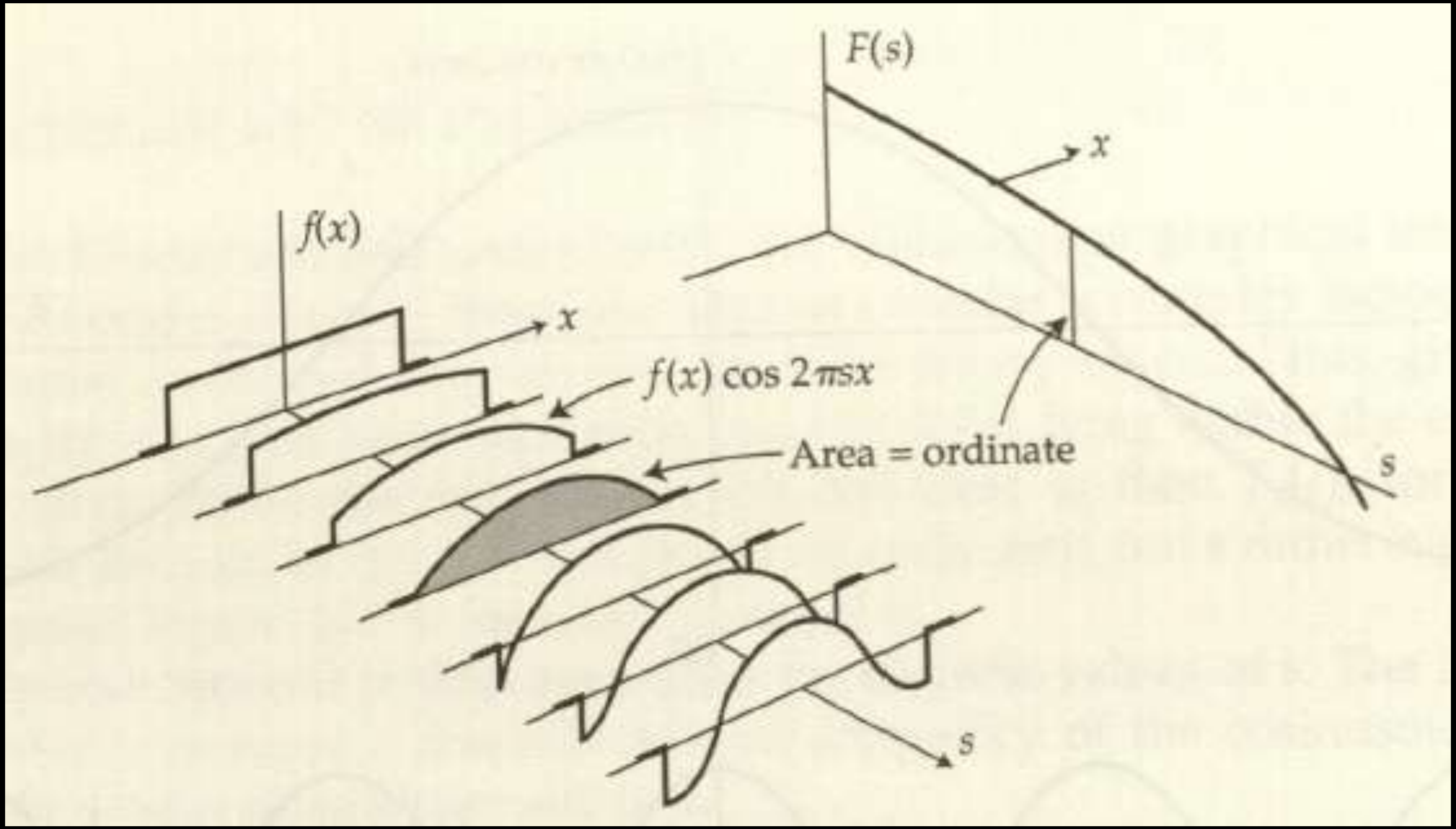
Real signals of interest have finite Δt time windows over the length of the signal, $N\Delta t$, so the limit can be dropped resulting in the approximate expression

$$X(f) \approx \sum_{m=0}^{N-1} x(m\Delta t)e^{-j2\pi f m\Delta t}\Delta t$$

The infinite number of really small subintervals can be replaced with a finite number of small subintervals.

***Careful this simple result may be deceiving*!** This result says that for a particular frequency f a sum must computed. If $X(f)$ is to be specified at every possible frequency then the sum must be evaluated at every possible frequency! That is an enormous amount of computation especially if N is large.

Illuminating example when $f(x)$ is a real & even Function



Source: Bracewell

$$f(x) \overset{\mathcal{F}}{\leftrightarrow} F(s)$$

Explanation

Some authors will say that the Continuous-Time Fourier Transform of a function $x(t)$ is the Continuous-Time Fourier Series of a function $x(t)$ in the limit as $T_0 \to \infty$. This is equivalent to saying the Fourier Series can be extended to aperiodic signals.

You can also think of the Fourier Transform as taking all the time amplitude information and mapping it into a single frequency. Here the “mapping” is multiplying the time signal by a complex exponential <u>at a particular frequency</u> and finding the area under the resulting curve.

Hence at <u>each</u> frequency this “mapping” or, in our case, sum must occur.

What is f exactly?

Much like in the case of sampled time signal we must settle for a sampled frequency signal

$$X[n] = X\left(n\frac{1}{N\Delta t}\right)$$

$\frac{1}{N\Delta t}$ can be thought of as the "distance" in frequency between each successive frequency sample. Compare it to Δt for the time sampled signal.

In light of the previous discussion we can say that $\frac{1}{N\Delta t}$ comes from mapping the entire period, $N\Delta t$, of $x(t)$ into a single frequency!

Frequency-Domain Housekeeping

So replacing f with $\frac{n}{N\Delta t}$ in our expression we have

$$X[n] = \sum_{m=0}^{N-1} x(m\Delta t)e^{-\frac{j2\pi nm}{\mathrm{T}}}\Delta t$$

With perhaps a slight abuse of notation we might say

$$x[m] \overset{\mathcal{F}}{\leftrightarrow} X[n]$$

What does it mean?

At each sampled frequency $f = \frac{n}{N\Delta t}$ (in this case let $n = \alpha$) there is an amplitude which is found by

$$X_{\alpha} = \Delta t \sum_{m=0}^{N-1} x(m\Delta t) e^{-\frac{j2\pi\alpha m}{\mathrm{T}}}$$

The evaluation of X_n at each frequency will populate the frequency domain with an approximation $\mathrm{X}(f)$ known as the Continuous-Time Fourier Transform of $x(t)$.

Give it a name!

$$\mathcal{DFT}\big(x(m\Delta t)\big) = \sum_{m=0}^{N-1} x(m\Delta t) e^{-\frac{j2\pi nm}{\mathrm{T}}}$$

This approximation is known as the Discrete Fourier Transform!

$$X\left(n\frac{1}{N\Delta t}\right) \cong \Delta t \cdot \mathcal{DFT}\big(x(m\Delta t)\big)$$

Note: The author of your textbook derives this result differently (Web Appendix H) under the condition that $|n| \ll N$.

Sources

Bracewell, R. (1999). *The Fourier Transform and Its Applications* (3rd ed.). Boston: McGraw Hill.

Lathi, B. (1992). *Linear Systems and Signals* (1st ed.). Carmichael, CA: Berkeley-Cambridge Press.

Stewart, J. (2007). *Calculus* (6th ed.). Belmont, CA: Cengage Learning.