Laplace Transforms

Lab 12 Notes

Unilateral Laplace Transform

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{+st} ds$$

This is a Contour Integral in the Complex Plane!

What is s exactly?

$$s = \sigma + j\omega$$

Typically we specify that $\sigma > 0$ this effectively specifies a region of convergence (ROC) for the transform.

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

This is effectively equivalent to the generalized Fourier Transform except we are keeping the σ part!

Region of Convergence

ROC are the s values for which the Laplace integral converges.

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-\sigma t}e^{-jwt}dt$$

Using Laplace transforms to Solve DEs

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = y(t), \qquad \mathrm{x}(0^-) = 0$$

$$\mathcal{L}\{\ddot{x}(t) + 3\dot{x}(t) + 2x(t)\} = \mathcal{L}\{y(t)\}$$

Look in table 8.3 and 8.4 in textbook

$$(s^2 + 3s + 2)X(s) = \mathrm{Y}(s)$$

$$H(s) = \frac{X(s)}{Y(s)} = \frac{1}{s^2 + 3s + 2}$$

Called a "transfer function". It transfers the input variable Y(s) to the output variable X(s).

let $y(t) = u(t)$

then $Y(s) = 1$ (that's a neat property!)

$$\therefore X(s) = \frac{1}{s^2 + 3s + 2}$$

But what is $x(t)$?

$$\therefore X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Values of s for which the denominator becomes zero are called "poles". Can you guess why?

Eww, Partial Fraction Decomposition! Isn't there an easier way?

Heaviside Cover-Up Method

Since the poles are both real and distinct (not repeated):

$$A = \lim_{s \to -1} \frac{1}{(s+1)(s+2)}(s+1) = 1$$

$$B = \lim_{s \to -2} \frac{1}{(s+1)(s+2)}(s+2) = -1$$

These values are called the residues corresponding to each pole.

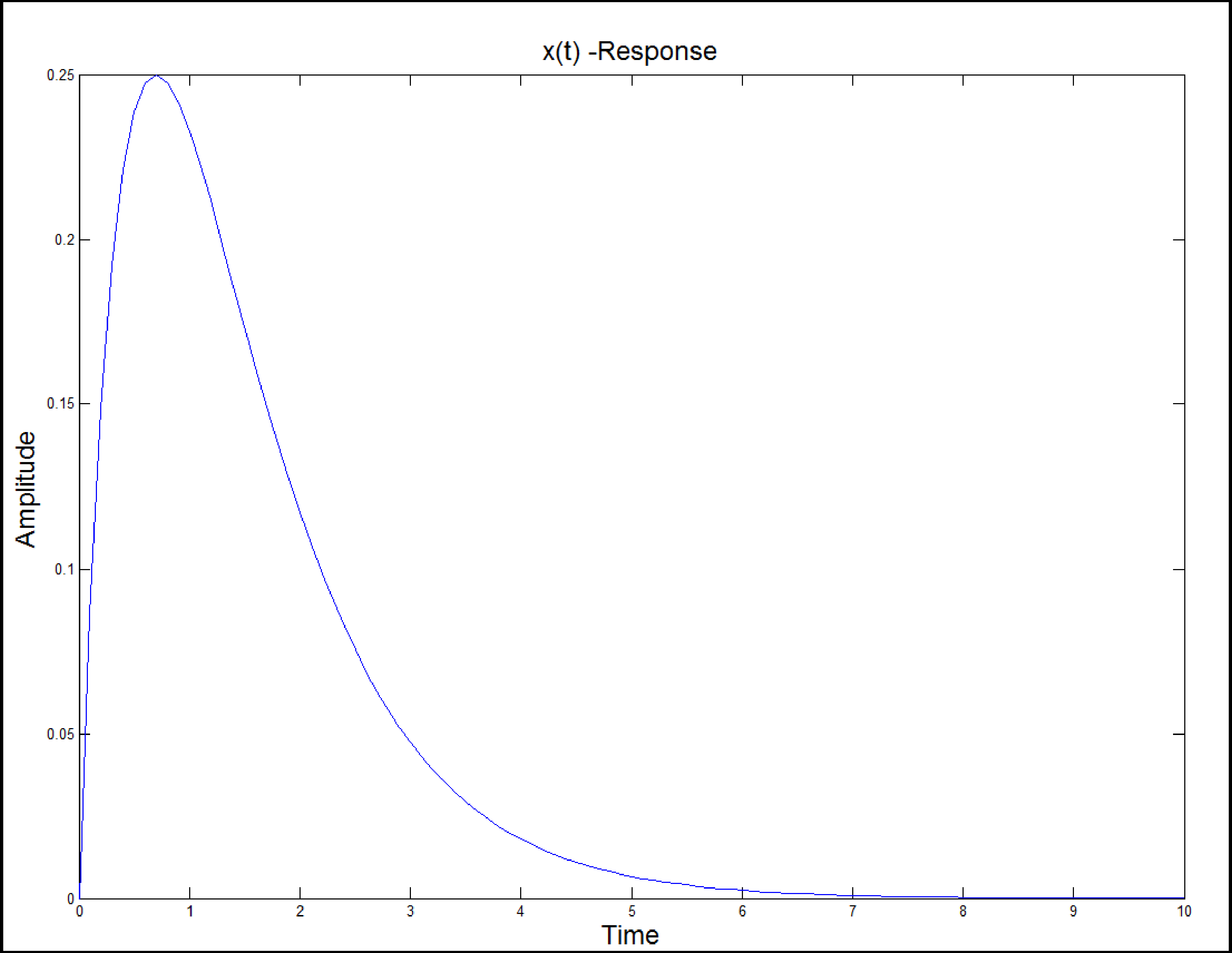
Inverse Laplace

$$X(s) = \frac{1}{s+1} + \frac{-1}{s+2}$$

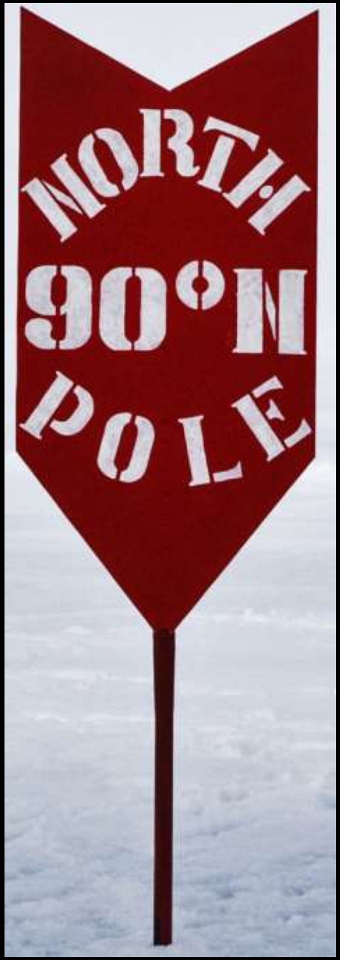
$$\mathrm{x}(t) = e^{-t} - \mathrm{e}^{-2\mathrm{t}}, \quad \mathrm{t} > 0$$

$$x(t) = ((1)e^{-1t} + (-1)e^{-2t})\, u(t)$$

Summary: The residues determined the amplitude of each component and poles determined the time constants of each component of the response!

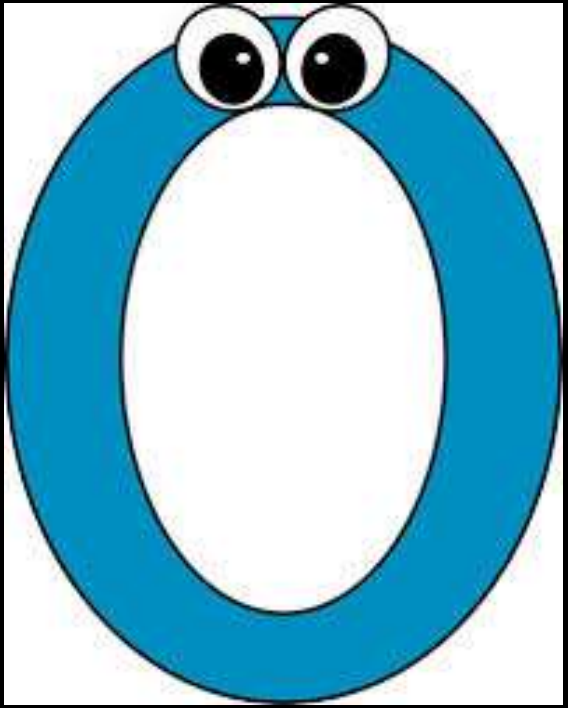

x(t) -Response
Amplitude
Time
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9
10

Poles! Zeros! Why do we care?



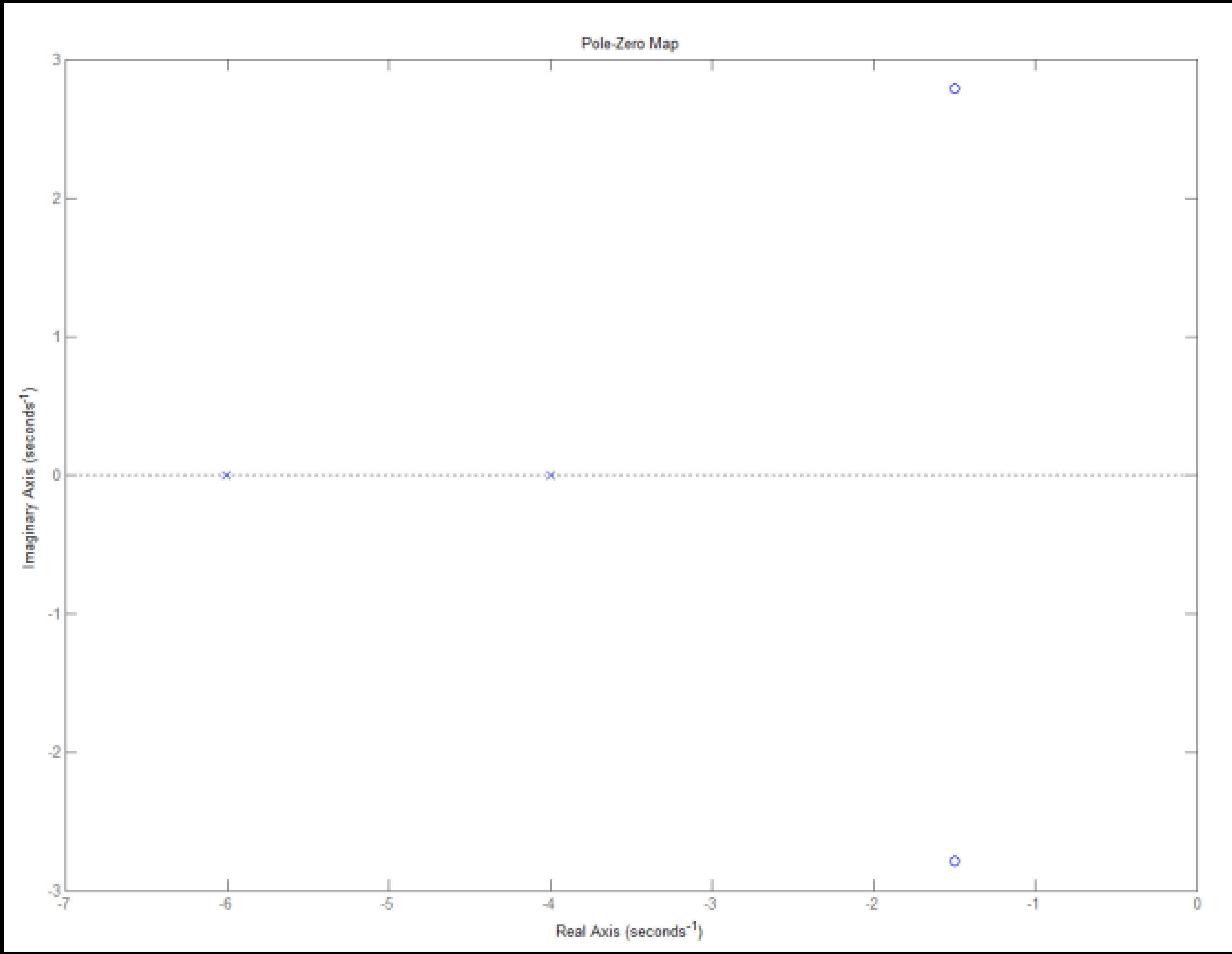
The placement of the Poles is crucial to the stability of any LTI system.

It turns out that any causal, stable LTI system must have its poles located in the left-half plane. This means that every one of the poles must have a negative real part!



Example

$$F(s) = \frac{s^2 + 3s + 10}{s^2 + 10s + 24}$$



ROC again

Region of Convergence is always the region to the right of all the *poles*.

Do you see why?

Matlab

roots()

Find the places where polynomial crosses the x-axis. Takes a matrix argument of the coefficients from highest to lowest order.

pzmap()

Plots the poles and zeros of any transfer function in the complex plane.

freqs()

Finds the frequency response from the transfer function.