Using Tangent Balls to Find Plane Sections of Natural Quadrics

James R. Miller
University of Kansas

Ronald N. Goldman
Rice University

Using solid geometry, we developed efficient and numerically robust algorithms to compute plane sections of the natural quadric surfaces common in modeling.

In computer-aided design and manufacturing, engineers and system developers often need to compute plane sections of geometric models. They use these sections to create tool path geometry for numerically-controlled machines or to compute various sectioned views in a drawing. Since portions of geometric models are often bounded by quadric surfaces, plane sections of quadrics are particularly important. Moreover, pairs of quadric surfaces sometimes intersect in planar curves, and system developers will frequently find it easiest to calculate these planar intersections by reducing the problem to one or two plane-quadric intersections.[1]

Algorithms for intersecting planes and quadric surfaces depend on the underlying representation scheme. Previous researchers like Levin and Sarraga have represented quadric surfaces by using implicit polynomial equations (the algebraic approach[2,3]), tagged sets of scalars, points, and vectors (the geometric approach[1,4,5]), or rational piecewise polynomial parameterizations (for example, nonuniform rational B-splines[6,7]). Algorithms for quadrics represented as implicit algebraic equations generally transform the quadric to a coordinate system in which the sectioning plane is the x-y plane, but if the quadric is expressed as a tagged set of scalars, points, and vectors, algorithms based on geometric constructions applied directly in untransformed world coordinates are more stable numerically. On the other hand, if the quadric is represented as a rational polynomial (or piecewise polynomial) parametric curve, we can easily represent the intersection as a curve in the parameter space of the quadric.

Researchers have developed few specialized techniques for computing plane sections of quadrics. We know of no techniques for the rational polynomial approach and only a few for representations based on implicit equations or tagged sets of geometric parameters. We developed robust algorithms, based solely on geometric constructions, that let us compute plane sections of the natural quadric surfaces when the surfaces are represented as tagged sets of scalars, points, and vectors.

The natural quadrics are the sphere, right circular cylinder, and right circular cone.[4] For simplicity, we use the terms "cylinder" for "right circular cylinder" and "cone" for "right circular cone." We derive the majority of our geometric constructions by applying a result about cylinders and cones from classical solid geometry.[8]

Although existing algorithms based on Levin's implicit-polynomial method[2] are more general than ours, serious numerical instabilities arise in practice. It is ironic that numerical instabilities are often the worst when relative geometric configurations are the simplest. Worse, it is precisely these simple configurations that designers often prefer. The general methods typically compute quantities whose sign is used to determine first whether a real intersection exists, and then the type of the intersection (circle, ellipse, hyperbola, and so forth). For example, if such a quantity is precisely zero, we get one type of result. If instead it is positive, we get another, and if negative, yet another. Since these quantities are the result of numerical computations, they will rarely be precisely zero. Generally speaking, the more calculations required to compute a quantity, the less reliable that quantity will be. Our methods determine the type of intersection with very few computations—in fact, often directly from the database representations with no additional computation. Moreover, the units are spatial; thus, we can easily determine reasonable tolerances for comparisons.

Previous work

We know of no comprehensive treatment of the problem of computing plane sections of quadric surfaces. Apparently, Levin[2] and Sarraga[3] used an algebraic approach based on coordinate system transformations. A general quadric in arbitrary position is represented algebraically as

$$Ax^2 + By^2 + Cz^2 + 2Dxy$$
$$+ 2Eyz + 2Fxz + 2Gx + 2Hy + 2Jz + K = 0$$

We can write this equation in matrix form:

$$pQp^T = 0$$

where

$$Q = \begin{pmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{pmatrix}$$

and

$$p = (x, y, z, 1)$$

Affine transformations map quadrics into other quadrics. If Q is a matrix describing an arbitrary quadric and R is an affine transformation matrix, we can compute the matrix Q' describing the quadric that results from applying R to Q as

$$Q' = R^{-1}Q(R^{-1})^T$$

In the algebraic approach to plane sections, we determine a combination of translations and rotations that map the sectioning plane to the x-y plane. We then apply these transformations to Q, that is, the algebraic representation of the quadric whose planar intersection we seek. We then drop all terms involving z in the transformed quadric representation. The resulting second degree equation in x and y describes the conic intersection in the transformed coordinate system.[2] Levin applies further transformations to bring the conic into canonical position in the x-y plane.[2]

There are at least two possible ways to represent this conic in the database, and it is not clear which, if either, Levin or Sarraga actually employed. You can store (1) the coefficients of the resulting second degree equation in x and y, and (2) the transformations that relate the local coordinate system to the world coordinate system. Alternatively, you can determine the type (ellipse, hyperbola, parabola) of the conic as well as its defining parameters (center, axes, dimensions) numerically. To calculate these geometric parameters in general position, you can then apply the inverse of the transformations used to bring the sectioning plane to the x-y plane.

The first approach is awkward since we would need additional coordinate system transformations every time we wished to use the representation. To test if a point lies on the conic, for example, we must first apply the transformations to the point. If the transformed z coordinate is 0 and the transformed x and y coordinates satisfy the conic's implicit equation, then the original point lies on the conic. Moreover, when we wish to draw the conic, we must generate a series of points in the local coordinate system and then transform each point to world coordinates by using the inverse of the stored transformations. The second approach doesn't require as many transformations, but it demands a considerable amount of computation as well as numerically sensitive tests to derive the type and geometric parameters describing the conic.

The primary advantage of the algebraic approach lies in its generality. You can compute plane sections of arbitrary quadric surfaces (not just the natural quadrics) using the algorithm described above without case-by-case analysis. This is precisely the disadvantage of our approach, since we develop a separate algorithm for each natural quadric. In our opinion, the advantages inherent in the geometric approach outweigh this disadvantage. From geometric considerations we derive the type of the conic as well as its defining parameters; hence we need no
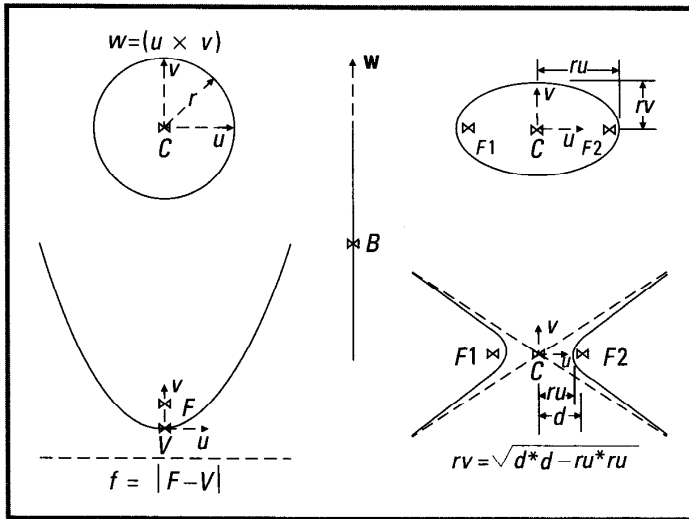
Figure 1. Geometric definitions of the conic sections.

conic section. However, these methods work for arbitrary rational surfaces and are therefore not specialized or optimized for quadrics. We assume that the surface is represented as

$$x = \frac{x(s, t)}{w(s, t)}, y = \frac{y(s, t)}{w(s, t)}, z = \frac{z(s, t)}{w(s, t)} \qquad (1)$$

where we can express the polynomials $x(s, t)$, $y(s, t)$, $z(s, t)$, and $w(s, t)$ in monomial, Bezier, B-spline, or any other convenient polynomial or piecewise polynomial basis.

The first method for rational polynomial surfaces begins as did the algebraic approach. We determine a series of transformations that map the sectioning plane onto the x-y plane. We then apply these transformations to Equation 1 to get

$$x^* = \frac{x^*(s, t)}{w^*w^*(s, t)}, y^* = \frac{y^*(s, t)}{w^*(s, t)}, z^* = \frac{z^*(s, t)}{w^*(s, t)}$$

The equation $z^*(s, t) = 0$ then represents the plane section in the parameter space of the rational surface. If the rational surface is a quadric, then the equation $z^*(s, t) = 0$ represents the conic in the parameter space of the quadric.

The second method substitutes the parametric equations of the surface into the implicit equation of the sectioning plane in general position. Given the implicit plane equation

$$ax + by + cz + d = 0$$

we substitute the expressions in Equation 1 for x, y, and z to again get a representation of the conic in the parameter space of the quadric:

$$ax(s, t) + by(s, t) + cz(s, t) + dw(s, t) = 0$$

But representations of intersection curves as implicit equations in parameter space (such as we get from either of the methods outlined above) are inconvenient. We must often intersect curves obtained from surface intersections with other curves or surfaces, and the intersection operation is more complex than necessary when we represent curves like conics in this way. In the boundary evaluation algorithm of solid modeling, we must be able to compare two curves for equality. When we represent them with implicit equations in the parameter space

numerical classification mechanism to determine the type of the conic from its algebraic coefficients. Furthermore, we need no transformations whatsoever, so we require far fewer calculations to arrive at a complete description of the conic sections. Fewer computations mean greater speed and improved numerical reliability. When expressions must be tested for equality, they are readily related to spatial quantities, making it easier to establish meaningful tolerances. Moreover, while our derivations are occasionally long, the algorithms we present are short. Therefore, the computer code required to implement these techniques is inexpensive.

Descriptions of geometric constructions for computing plane sections are as rare in the literature as those for the algebraic method. Rogers and Adams described special techniques for certain types of plane-sphere sectioning operations.[9] Miller described how to intersect planes and spheres in the general case using geometric constructions,[10] but he didn't discuss the computation of plane sections for other types of quadrics. The plane-sphere algorithm Miller presented[10] is the same as that given under "Plane-sphere intersection" below.

To the best of our knowledge, no one has published descriptions of plane-section algorithms specialized for quadrics represented by rational polynomial parameterizations. We briefly mention here two possible approaches for finding plane sections of rational parametric surfaces. If the surfaces happen to represent quadrics, then you can use these methods to get a description of the resulting

Table 1. Geometric descriptions of lines and conics.		
Curve	Description of geometric parameters	
Line	(base point, direction vector)	$(B, \mathbf{w})$
Circle	(center, normal to the plane containing the circle, radius)	$(C, \mathbf{w}, r)$
Ellipse	(center, major axis, minor axis, major radius, minor radius)	$(C, \mathbf{u}, \mathbf{v}, r_u, r_v)$
Parabola	(vertex, directrix vector, focus vector, focal length)	$(V, \mathbf{u}, \mathbf{v}, f)$
Hyperbola	(center, major axis, minor axis, major radius, minor radius)	$(C, \mathbf{u}, \mathbf{v}, r_u, r_v)$

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of rational surfaces, this test is extremely complex. Finally, many design and manufacturing applications are especially easy to perform and reliable in operation when supplied with direct representations of curves such as circles. Representing conics as implicit equations in parameter space complicates these operations and ultimately makes them less reliable.

In this article, we develop geometric constructions for computing plane sections of the natural quadric surfaces that operate on planes and natural quadrics in general position and orientation. These procedures generate geometric descriptions of the resulting conic sections. Our algorithms are extremely fast, numerically robust, and do not employ coordinate system transformations of any sort.

Geometric notation and tools

Geometric representations of conics and quadrics are generally characterized by a local coordinate system with associated scalar parameters. The local coordinate system is defined by

1. Three mutually perpendicular unit vectors ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) that describe the orientation of the conic or quadric and
2. A base point O that fixes the position of the curve or surface in space. The scalar parameters determine the size of the conic or quadric.

We adopt certain conventions on the use of the vectors ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$). We use the vectors ($\mathbf{u}$, $\mathbf{v}$), for example, to specify 2D orientations in a plane (for example, the major and minor axis vectors for an ellipse). We use the vector $\mathbf{w}$ for line directions, plane normals, and cylinder and cone axis vectors.

Obviously there is redundant information in the complete coordinate system, and we need to specify only portions of it to determine uniquely the position and orientation of a particular conic or quadric. We summarize the geometric parameters that uniquely define lines and conics in Table 1 and illustrate them in Figure 1. We list those defining planes and natural quadrics in Table 2 and illustrate them in Figure 2. Throughout this article, we assume that the vectors associated with the geometric representations are unit vectors.

Using vector techniques, we can easily derive both parametric and implicit representations of these second degree curves and surfaces from their geometric representations. For example, a rational polynomial parameterization for the ellipse is given by
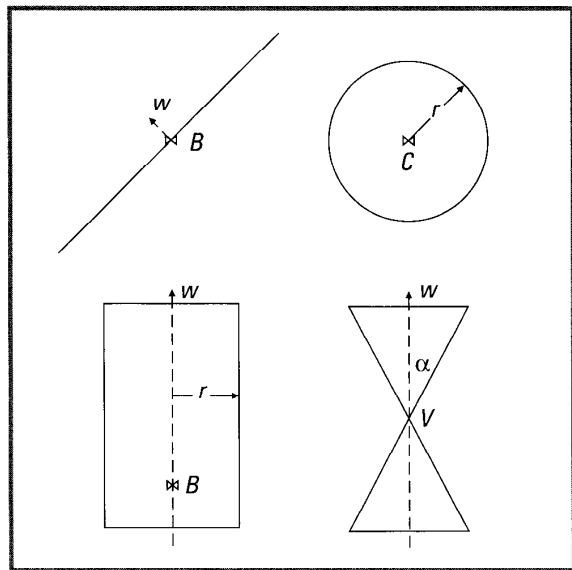

Figure 2. Geometric definitions of the natural quadric surfaces.

$$E(t) = C + \frac{r_u(1 - t^2)}{1 + t^2}\mathbf{u} + \frac{2r_v t}{1 + t^2}\mathbf{v}$$

and an implicit equation for the cylinder is simply

$$(P - B) \cdot (P - B) - [(P - B) \cdot \mathbf{w}]^2 - r^2 = 0$$

where P is an arbitrary point in 3-space.

An advantage of the geometric representation is this flexibility; we can robustly derive these other two common representations. Moreover, all the data in the geometric representation has a clear-cut physical meaning. This is not true for, say, the coefficients of the implicit polynomial representation.

In the sequel, we assume that certain primitive functions create and manipulate scalars, points, vectors, curves, and surfaces. We briefly summarize some of them below.

The function Line(Q, $\mathbf{v}$) returns a line whose base point and unit direction vector are as specified. The function Normalize($\mathbf{v}$) returns a unit vector whose direction is the same as that of $\mathbf{v}$. The function Signed_distance_from_plane(Q, P) calculates the signed distance as

$$((Q - P.B) \cdot P.\mathbf{w})$$

Based on the Signed_distance_from_plane(Q, P) function, we assume we can test reliably (that is, to within some prespecified tolerance) whether a point is on a plane by checking for a zero-signed distance.
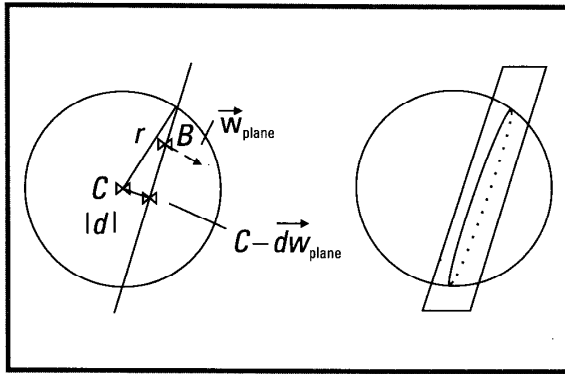
Table 2. Geometric descriptions of planes and natural quadric surfaces.		
Surface	Description of geometric parameters	Notation
Plane	(base point, normal vector)	(B, $\mathbf{w}$)
Sphere	(center, radius)	(C, r)
Cylinder	(base point, axis vector, radius)	(B, $\mathbf{w}$, r)
Cone	(vertex, axis vector, half-angle)	(V, $\mathbf{w}$, α)

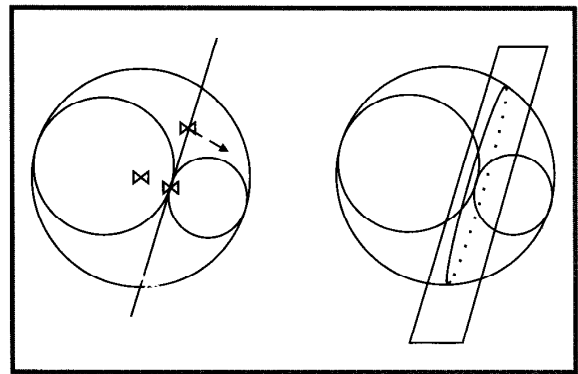**Figure 3. The plane intersects the sphere in a circle.**

Figure 4. Two inscribed tangent spheres touching at the center of the circle.

Plane-quadric intersections

When planes intersect quadrics, the result is either a (possibly degenerate) conic section or a single point. In this section we describe algorithms for intersecting planes with spheres, cylinders, and cones.

As described in Table 2, the input to the algorithms presented in this section is the geometric parameters of a plane and a natural quadric. The output is the defining geometric parameters of the (possibly degenerate) conic as described in Table 1 or the coordinates of the single point of intersection.

There are many ways to compute plane-quadric intersections geometrically. Any simplification, enhancement, or improvement to these algorithms automatically improves the operation of routines that depend on them (such as those described in our other article[1]).

First we treat the simplest case, namely plane-sphere intersections. While we won't need the tangent ball technique on which the rest of this article is based, we observe an analogous result that serves as a preview of how we'll use the tangent ball technique for cylinders and cones. Next we present algorithms for plane-cylinder intersections. We begin by introducing the tangent ball theorem for plane sections of cylinders. We then demonstrate how to apply the theorem to derive a robust plane-cylinder intersection algorithm based on geometric constructions. Finally we treat the plane-cone case. Again we begin by stating the tangent ball theorem for cones, then demonstrate how we can use it to derive a plane-cone intersection algorithm based purely on geometric constructions.

We summarize the derivations of each section with pseudocode implementations of intersection algorithms. These algorithms occasionally require equality tests which, for clarity of exposition, are written as "if $x = y$ then. . . ." In practice you test for equality within some tolerance, the value of which depends on the quantities being examined. In our algorithms, the quantities are always either distances or angles; hence, you can compute reasonable tolerances based on, for example, the overall size of the model.

Plane-sphere intersection

When a plane intersects a sphere, the result is either a single tangent point or a circle. The distance from the sphere center to the plane is the determining factor. The following algorithm suffices (see Figure 3):

input: P: plane; S: sphere
d : = signed_distance_from_plane(S.C,P)
if abs(d) = S.r **then**
  output: tangent point : = S.C − d * P.**w**
else if abs(d) > S.r **then**
  output: no intersection
else
output: circle: C : = S.C − d * P.**w**
      r: = sqrt(S.r^2 − d^2)
      w: = P.**w**

Notice that the two spheres tangent to both *P* and *S* are tangent to *P* at the center of the circle of intersection (Figure 4). As we see later, an analogous result holds for plane-cylinder and plane-cone intersections. That is, we can construct a pair of spheres, each tangent to the cylinder or cone in a circle and tangent to the plane at a point. The tangent points on the plane are the foci of the conic of intersection, and we use this fact to derive algorithms for constructing the geometric parameters of the intersection conic.

Plane-cylinder intersection

The intersection of a plane and a cylinder is either empty, a single tangent line, two lines, an ellipse, or a circle. We call the acute angle between the cylinder axis and the plane normal θ.
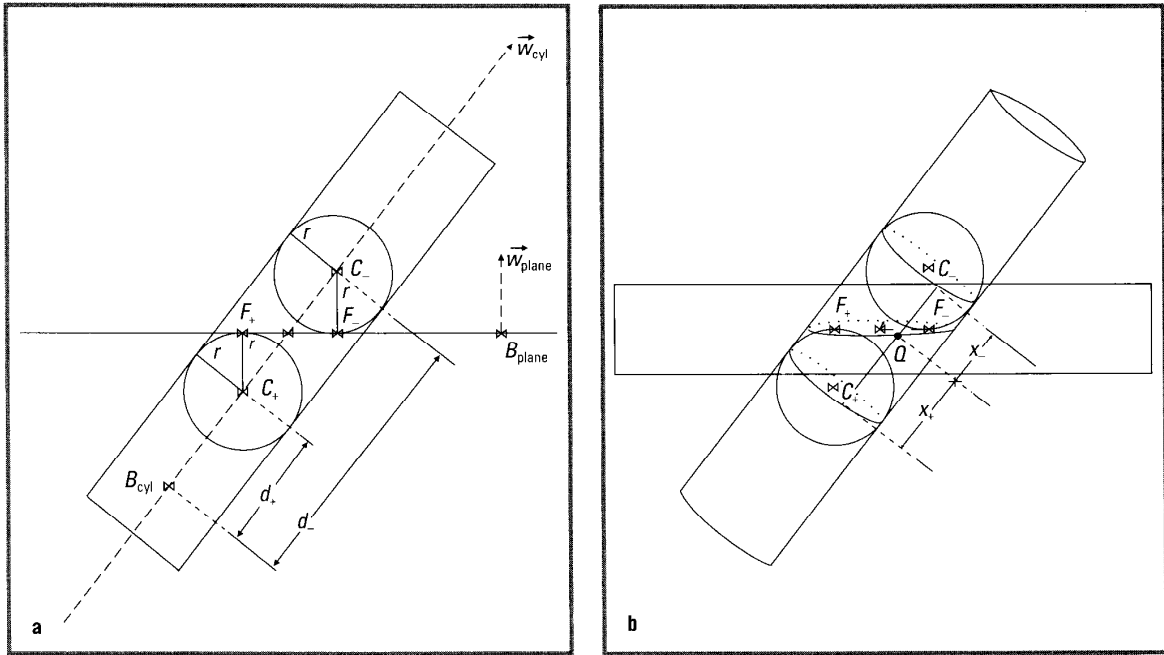
Figure 5. (a) The plane intersects the cylinder in an ellipse. (b) Off-axis view of the geometry of (a).

One or two lines can occur only if $\theta = \pi/2$. In this case, the distance between the cylinder axis and the plane (equivalently, the distance between the cylinder base point and the plane) determines whether the intersection is empty or one or two lines.

The intersection cannot be empty when $\theta \neq \pi/2$. It will then, in general, be an ellipse. If $\theta = 0$, the intersection is actually a circle. We can detect this after computing the ellipse parameters by testing for equality of the major and minor radii. It is more efficient and numerically reliable, however, to test for this configuration at the start of the algorithm and to treat the circle intersection as a special case. The parameters of the circle are easy to compute. We use this approach in the algorithm below.

If we did not treat this as a special case, not only is $r_u = r_v$, but also **u** and **v** as calculated below are zero vectors. Since these vectors are not required for circles (see Table 1), this is not a serious problem. But note that this means we have redundant conditions signalling a circle intersection:

1. $r_u = r_v$,
2. $\mathbf{u} = (0, 0, 0)$, and
3. $\mathbf{v} = (0, 0, 0)$.

Redundancy invites inconsistency, especially in numerical computations. For example, the vagaries of floating-point arith-metic can lead to a situation where the first two equations are deemed to be false, but the third is true.

When θ does not equal 0 or 90 degrees, the intersection is an ellipse. We derive an efficient and robust algorithm for this case by applying the following theorem from solid geometry.

The Tangent Ball Theorem for Cylinders: When the intersec-tion of a plane and a cylinder is an ellipse, there are precisely two spheres that are tangent to the cylinder in a circle and tangent to the plane at a point. The two points on the plane at which these spheres are tangent are the foci of the ellipse. Fur-thermore, the distance along a cylinder ruling between the two tangent circles is twice the major radius of the ellipse.

Sketch of Proof: Though this result is well-known in solid geometry,[8] for completeness we sketch a proof. Consider the geometry of Figure 5, in which a plane cuts a cylinder in an ellipse. For any point Q on the ellipse, we define x_+ as the distance along a ruling between Q and the circle centered at C_+. Similarly we define x_- as the distance along a ruling between Q and the circle centered at C_-. (The significance of the "+" and "−" subscripts becomes clear shortly.) Since the distance along a cylinder ruling between the two circles is a constant, the sum $(x_+ + x_-)$ is this constant distance and is therefore independent of Q. But note that the line passing through Q and F_+ is also tangent to the sphere; hence the distance between Q and F_+ must be x_+. Similarly, the distance between Q and F_- must be x_-. Since Q is an arbitrary point on the ellipse, we see that the ellipse can be characterized as the set of points on the plane, the
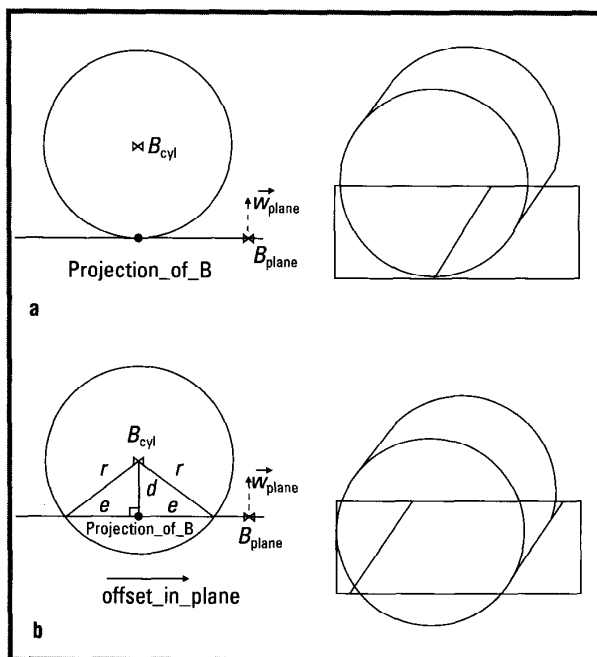
Figure 6. (a) The plane is tangent to the cylinder along a line. (b) The plane cuts the cylinder in two distinct lines.

sum of whose distances from F_+ and F_- is a constant. But this means that F_+ and F_- must be the foci of the ellipse, and the constant $(x_+ + x_-)$ must be twice the major radius. This concludes the proof.

We can apply this theorem to derive explicit expressions for the foci. We can further use these expressions to derive the various geometric parameters, describing the ellipse in general position and orientation. We assume the cylinder is defined by $(B_{cyl}, \mathbf{w}_{cyl}, r)$, and the plane is defined by $(B_{plane}, \mathbf{w}_{plane})$.

The centers of the two tangent spheres must lie on the cylinder axis, and their radii must of course be r. Clearly the two spheres lie on opposite sides of the plane (see Figure 5). If the center of a sphere is C, then the point of tangency on the plane is $C \pm r\mathbf{w}_{plane}$. The signed distances along the axis line from the base point of the cylinder to these two centers is denoted d_+ and d_-. Thus

$$C_+ = B_{cyl} + d_+\mathbf{w}_{cyl}$$
$$C_- = B_{cyl} + d_-\mathbf{w}_{cyl}$$

For each center point, we construct the resulting tangent point on the plane (that is, a focus of the ellipse) as

$$F_+ = B_{cyl} + d_+\mathbf{w}_{cyl} + r\mathbf{w}_{plane}$$
$$F_- = B_{cyl} + d_-\mathbf{w}_{cyl} - r\mathbf{w}_{plane}$$

To determine d_+ and d_-, we substitute these expressions for the foci into the implicit equation of the plane:

$$((B_{cyl} + d_+\mathbf{w}_{cyl} + r\mathbf{w}_{plane}) - B_{plane}) \cdot \mathbf{w}_{plane} = 0$$
$$((B_{cyl} + d_-\mathbf{w}_{cyl} - r\mathbf{w}_{plane}) - B_{plane}) \cdot \mathbf{w}_{plane} = 0$$

Solving these equations for d_+ and d_-,

$$d_+ = \frac{(B_{plane} - B_{cyl}) \cdot \mathbf{w}_{plane} - r}{\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}}$$

$$d_- = \frac{(B_{plane} - B_{cyl}) \cdot \mathbf{w}_{plane} + r}{\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}}$$

Notice that when $(\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}) > 0$, $d_- > d_+$, and when $(\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}) < 0$, $d_- < d_+$.

Since the center of the ellipse is the midpoint of the foci, we have

$$C = \frac{F_+ + F_-}{2} = B_{cyl} + \frac{d_+ + d_-}{2}\mathbf{w}_{cyl}$$
$$= B_{cyl} + \frac{(B_{plane} - B_{cyl}) \cdot \mathbf{w}_{plane}}{\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}}\mathbf{w}_{cyl}$$

Examining this formula, we clearly see that the ellipse's center lies on the axis of the cylinder, and hence the center is the intersection of the axis line with the plane. We can then find the direction of the major axis vector as

$$F_+ - F_- = (d_+ - d_-)\mathbf{w}_{cyl} + 2r\mathbf{w}_{plane} = \frac{-2r}{\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}}\mathbf{w}_{cyl} + 2r\mathbf{w}_{plane}$$

For now we are only interested in the direction, so we can write

$$\frac{\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}}{-2r}(F_+ - F_-) = \mathbf{w}_{cyl} - (\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane})\mathbf{w}_{plane}$$

Therefore, we get the major axis direction from the component of $\mathbf{w}_{cyl}$ perpendicular to $\mathbf{w}_{plane}$. By the Tangent Ball Theorem, the distance along a ruling from one tangent circle to the other is twice the major radius. Clearly from Figure 5 this distance is $|d_- - d_+|$. Hence

$$r_u = \frac{|d_- - d_+|}{2} = \frac{r}{|\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}|}$$

Finally, to compute the minor radius, we find the distance c between the center and a focus and then compute the minor radius r_v as $\sqrt{(r_u^2 - c^2)}$. Since c is also half the distance between the foci, we first calculate the square of the distance between the foci.

$$4c^2 = |F_+ - F_-|^2 = 4r^2 \left(\frac{1}{(\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane})^2} - 1 \right)$$

Since $(\mathbf{w}_{cyl} \cdot \mathbf{w}_{plane}) = \cos \theta$, we simplify this expression to get

$$2c = |F_+ - F_-| = \sqrt{4r^2 \left(\frac{1}{\cos^2 \theta} - 1 \right)} = 2r \tan \theta$$

Then we compute r_v as

$$r_v = \sqrt{r_u^2 - c^2} = \sqrt{\left(\frac{r}{\cos \theta} \right)^2 - (r \tan \theta)^2} = \sqrt{r^2 \frac{1 - \sin^2 \theta}{\cos^2 \theta}} = r$$

Thus the minor radius of the ellipse is the same as the cylinder radius r.

The following algorithm summarizes the treatment of plane-cylinder intersections. Note that we never actually compute the foci in the algorithm. We simply used the formulas describing them to derive compact expressions for the ellipse parameters summarized in Table 1. Though our derivation was a bit long and tedious, notice how compact the final code becomes.

```
input: P: plane; C: cylinder
d : = signed_distance_from_plane(C.B,P)
Projection_of_B : = C.B - d * P.w
cos_theta = C.w · P.w
abs_cos_theta = abs(cos_theta)
if cos_theta = 0 then
    {
    intersection is empty or consists of one or two lines; see Figure 6
    }
    if abs(d) = C.r then
        output: tangent line: B : = Projection_of_B
                          w : = C.w
    else if abs(d) > C.r then
        output: no intersection
    else { two lines }
        offset_in_plane : = C.w × P.w
        e : = sqrt(C.r² - d²)
        output: line 1: B := Projection_of_B - e * offset_in_plane
                        w : = C.w
                line 2: B : = Projection_of_B + e * offset_in_plane
                        w : = C.w
else { ellipse or circle }
    if abs_cos_theta = 1 then { See Figure 7 }
        output: circle: C : = Projection_of_B
                        w: = P.w
                        r: = C.r
    else { refer to Figure 5 }
        output: ellipse: C : = C.B - (d / cos_theta)*C.w
                u: = normalize(C.w - cos_theta * P.w)
                v: = P.w × u
                ru: = C.r / abs_cos_theta
                rv: = C.r
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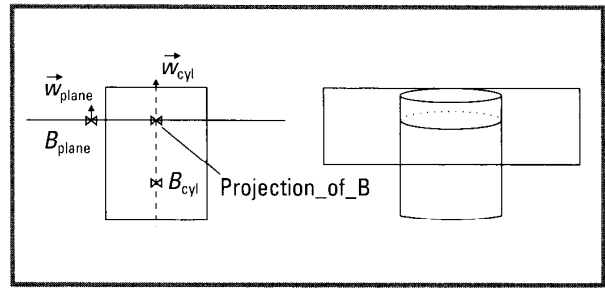

Figure 7. The plane intersects the cylinder in a circle.

Plane-cone intersection

A plane and a cone always intersect, even if just at the vertex of the cone. Any conic section is possible depending on the angle between the cone axis and the plane normal. Before we derive the algorithms, let's look at the theorem on which we based our approach.[8]

The Tangent Ball Theorem for Cones: When the intersection of a plane and a cone is a nondegenerate conic section, there are precisely one or two spheres that are tangent to the cone in a circle and tangent to the plane at a point. If there is only one such sphere, the intersection is a parabola, and the point on the plane at which the sphere is tangent is the focus of the parabola. If there are two such spheres, the intersection is an ellipse or a hyperbola, and the two points on the plane at which the spheres are tangent are the foci. The distance along a cone ruling between the two tangent circles is twice the major radius of the ellipse or hyperbola.

Sketch of Proof: Again, this result is well-known in solid geometry,[8] but for completeness we sketch a proof. The proof is analogous to that for the cylinder version of the theorem. We sketch the proof of this version assuming an ellipse intersection.

Consider the geometry of Figure 9, in which a plane cuts a cone in an ellipse. For any point Q on the ellipse, we define x_1 as the distance along a ruling between Q and the tangent circle determined by the sphere centered at C_1. Similarly, we define x_2 as the distance along a ruling between Q and the tangent circle determined by the sphere centered at C_2. Since the distance along a cone ruling between the two circles is a constant, the sum $(x_1 + x_2)$ is this constant distance and is therefore independent of Q. But note that the line passing through Q and F_1 is also tangent to the sphere centered at C_1. Hence the distance between Q and F_1 must be x_1. Similarly, the distance between Q and F_2 must be x_2. Since Q is an arbitrary point on the ellipse, we see that the ellipse can be characterized as the set of points on the plane, the sum of whose distances from F_1 and F_2 is a constant. But this means that F_1 and F_2 must be the foci of the ellipse, and the constant $(x_1 + x_2)$ must be twice the major radius. This concludes the proof.

We assume that $(V, \alpha, \mathbf{w}_{cone})$ defines the cone, and $(B, \mathbf{w}_{plane})$ defines the plane. We can apply this theorem to derive explicit expressions for the foci, and we can then use these expressions to derive the various geometric parameters describing the conic in general position and orientation. As with plane-cylinder intersections, we never need to compute the foci in the final computer code. Instead, we use the expressions defining them to derive compact formulas for the defining geometric parameters of the conic, as summarized in Table 1.

We distinguish here two primary cases according to whether or not the vertex of the cone lies on the plane. If the cone vertex lies on the plane, then there are no spheres such as those described in the Tangent Ball Theorem For Cones. We must compute the intersection for this case (vertex only, one tangent line, or two distinct lines) by another method. First let's consider the case in which the vertex of the cone doesn't lie on the plane. We then show how to treat the configuration in which the vertex lies on the plane as a limiting case.

When the cone vertex is not on the plane

If the cone vertex doesn't lie on the plane, then the intersection is a nondegenerate conic section, and we can pursue the tangent ball approach. Without loss of generality, we make the following simplifying assumptions:

$$((V - B) \cdot \mathbf{w}_{plane}) < 0 \quad \text{(if not, replace } \mathbf{w}_{plane} \text{ with } -\mathbf{w}_{plane})$$
$$(\mathbf{w}_{cone} \cdot \mathbf{w}_{plane}) \geq 0 \quad \text{(if not, replace } \mathbf{w}_{cone} \text{ with } -\mathbf{w}_{cone}) \quad (1)$$

We call the angle between the cone axis vector and the plane normal θ. Therefore $\cos \theta = \mathbf{w}_{cone} \cdot \mathbf{w}_{plane}$.

The condition for a sphere (C, r) to be tangent to the cone in a circle is[1]

$$C = V + \frac{r}{\sin \alpha} \mathbf{w}_{cone} \quad (2)$$

We take some liberty with r in Equation 2, letting it be negative. We understand the radius of the sphere to be $|r|$, and we let the sign of r place the sphere on a particular half of the cone. The condition for a sphere to be tangent to the plane is

$$|(C - B) \cdot \mathbf{w}_{plane}| = |r|$$
$$\therefore ((C - B) \cdot \mathbf{w}_{plane})^2 = r^2 \quad (3)$$

Substituting the expression for C from Equation 2 into Equation 3, we find

$$\left(\left((V - B) + \frac{r}{\sin \alpha} \mathbf{w}_{cone} \right) \cdot \mathbf{w}_{plane} \right)^2 = r^2$$

When we expand and gather terms of like powers in r, we get

$$r^2 \left(\frac{(\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})^2}{\sin^2 \alpha} - 1 \right) + r \left(\frac{2((V - B) \cdot \mathbf{w}_{plane})(\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})}{\sin \alpha} \right) + ((V - B) \cdot \mathbf{w}_{plane})^2 = 0 \quad (4)$$

Observe that the discriminant of Equation 4 is a perfect square:

$$\mathrm{disc} = \frac{4((V - B) \cdot \mathbf{w}_{plane})^2 (\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})^2}{\sin^2 \alpha}$$
$$- 4 \left(\frac{(\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})^2}{\sin^2 \alpha} - 1 \right) ((V - B) \cdot \mathbf{w}_{plane})^2$$
$$= 4((V - B) \cdot \mathbf{w}_{plane})^2$$
$$\therefore \sqrt{\mathrm{disc}} - 2((V - B) \cdot \mathbf{w}_{plane})$$

We can therefore write expressions for the two possible values of r:

$$r_1 = \frac{\dfrac{-2((V - B) \cdot \mathbf{w}_{plane})(\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})}{\sin \alpha} - 2((V - B) \cdot \mathbf{w}_{plane})}{2 \left(\dfrac{(\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})^2}{\sin^2 \alpha} - 1 \right)}$$
$$= \frac{((V - B \cdot \mathbf{w}_{plane}) \sin \alpha ((\mathbf{w}_{cone} \cdot \mathbf{w}_{plane}) + \sin \alpha)}{\sin^2 \alpha - (\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})^2}$$
$$= \frac{((V - B) \cdot \mathbf{w}_{plane}) \sin \alpha}{\sin \alpha - (\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})}$$
$$= \frac{((V - B \cdot \mathbf{w}_{plane}) \sin \alpha}{\sin \alpha - \cos \theta} \quad (5)$$

Similarly

$$r_2 = -\frac{((V - B) \cdot \mathbf{w}_{plane}) \sin \alpha}{\sin \alpha + \cos \theta} \quad (6)$$

Since our cones are nondegenerate (that is, $\alpha > 0$) and since by assumption $\cos \theta = (\mathbf{w}_{cone} \cdot \mathbf{w}_{plane}) \geq 0$, r_2 is always well defined. The denominator of Equation 5 will be zero, however, if the acute angle between the cone axis line and the plane is α.

We consider three primary subcases according to whether $\cos \theta = (\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})$ is equal to, greater than, or less than $\sin \alpha$. These cases lead respectively to parabola, ellipse, and hyperbola intersections.

Parabola: $\cos \theta = \sin \alpha$

If $\cos \theta = \sin \alpha$, then $\theta + \alpha = \pi / 2$, the acute angle between the cone axis line and the plane is α, and the plane intersects one half of the cone in a parabola (see Figure 8a). Only one tangent ball exists, because the denominator of Equation 5 is zero in this case. We drop the subscript "2" from r_2 and refer to the radius of this sphere as r. This ball is tangent to the plane at the focus of the parabola (see Figure 8). Now we can find the vertex of the parabola directly.

Since $\cos \theta = \sin \alpha$, $r = -((V - B) \cdot \mathbf{w}_{plane}) / 2 = d / 2$ where d is the distance from the vertex of the cone to the plane. Referring to Figure 8a, we see that $\beta = \theta - \alpha = \pi / 2 - 2\alpha$. Therefore the distance
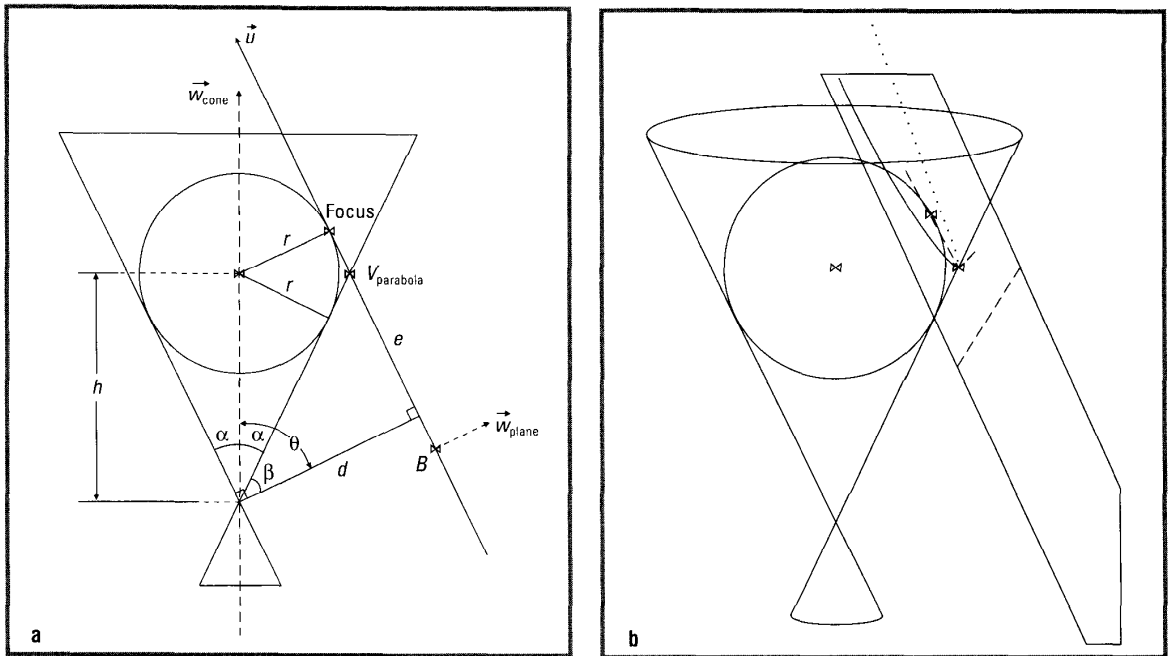
Figure 8. (a) The plane intersects the cone in a parabola. Only one tangent sphere exists. (b) Off-axis view of the geometry of 8a.

$$e = d \tan\left(\frac{\pi}{2} - 2\alpha\right) = d \cot(2\alpha) = d \cot(\pi - 2\theta)$$

$$= -d \cot(2\theta) = d\left(\frac{\tan\theta - \cot\theta}{2}\right)$$

and the distance

$$h = \frac{r}{\sin\alpha} = \frac{d}{2\cos\theta}$$

By symmetry the vertex and focus of the parabola must lie in the plane containing the cone vertex, $\mathbf{w}_{cone}$, and $\mathbf{w}_{plane}$. The unit focal direction vector $\mathbf{v}$ of the parabola must lie in this plane as well, and it must be perpendicular to $\mathbf{w}_{plane}$. The focal direction is therefore the component of $\mathbf{w}_{cone}$ perpendicular to $\mathbf{w}_{plane}$.

$$\mathbf{v} = \text{Normalize}(\mathbf{w}_{cone} - (\mathbf{w}_{cone} \cdot \mathbf{w}_{plane})\mathbf{w}_{plane})$$
$$= \text{Normalize}(\mathbf{w}_{cone} - \cos\theta\mathbf{w}_{plane}) \tag{7}$$

We can thus construct the vertex and focus of the parabola as

$$V_{parabola} = V_{cone} + d\mathbf{w}_{plane} + e\mathbf{v}$$
$$\text{Focus} = V_{cone} + h\mathbf{w}_{cone} + r\mathbf{w}_{plane}$$

Observe that the vector (Focus − $V_{parabola}$) must be parallel to $\mathbf{v}$. Furthermore, its length is, by definition, the focal length of the parabola. We now use these facts to derive a compact formula for the focal length of the parabola. First

$$(\text{Focus} - V_{parabola}) = h\mathbf{w}_{cone} + (r - d)\mathbf{w}_{plane} - e\mathbf{v}$$

$$= \frac{d}{2}\left[\frac{1}{\cos\theta}(\mathbf{w}_{cone} - \cos\theta\mathbf{w}_{plane}) + (\cot\theta - \tan\theta)\mathbf{v}\right]$$

But clearly

$$|\mathbf{w}_{cone} - \cos\theta\mathbf{w}_{plane}| = \sqrt{1 - 2\cos^2\theta + \cos^2\theta} = \sin\theta$$

Thus, from Equation 7 we see that $\mathbf{w}_{cone} - \cos\theta\mathbf{w}_{plane} = \sin\theta\mathbf{v}$, and we can write

$$(\text{Focus} - V_{parabola}) = \frac{d}{2}[\tan\theta\mathbf{v} + (\cot\theta - \tan\theta)\mathbf{v}] = \frac{d}{2}\cot\theta\mathbf{v}$$

Therefore the focal length of the parabola is $(d \cot\theta)/2$. We summarize these results in the following algorithm (see Figure 8). We assume here that the conditions in Equation 1 are satisfied.

```
input: P: plane; C: cone
d : = (P.B – C.V) · P.w
cos_theta : = P.w · C.w
sin_theta : = sqrt(1.0 - cos_theta * cos_theta)
tan_theta : = sin_theta / cos_theta
cot_theta : = 1.0 / tan_theta
e : = 0.5 * d * (tan_theta – cot_theta)
focus_vector : = normalize(C.w – cos_theta * P.w)
output: parabola: V: = C.V + d * P.w + e * focus_vector
```
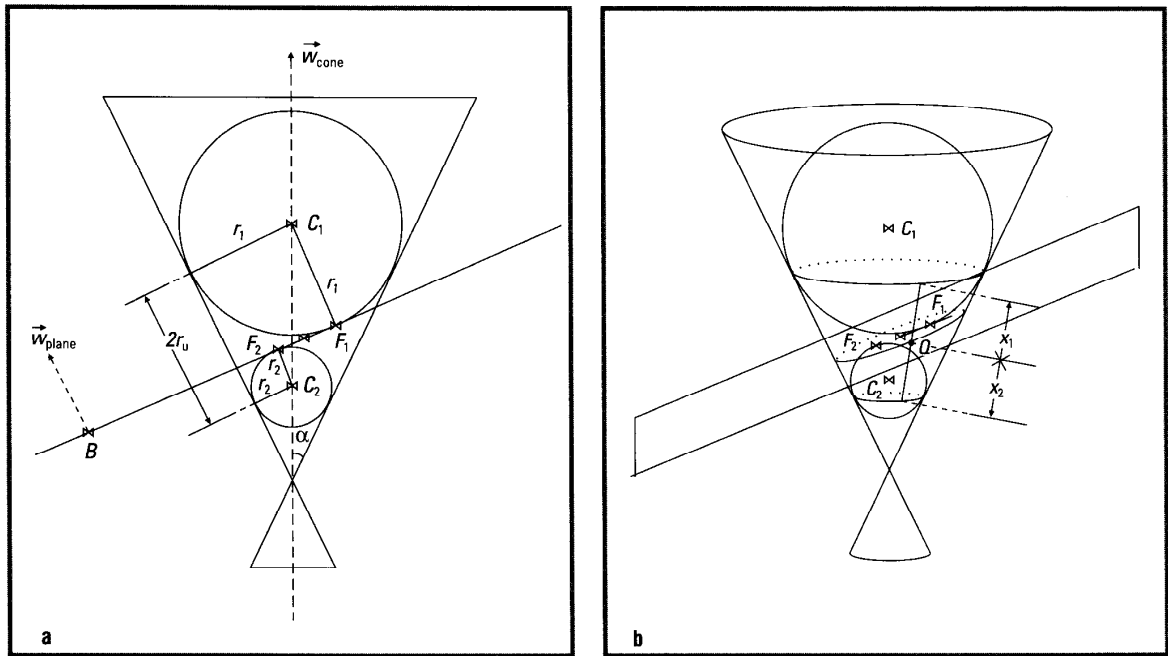
Figure 9. (a) The plane intersects the cone in an ellipse. (b) Off-axis view of the geometry of 9a.

$\mathbf{u}:$ = focus_vector $\times$ P.w
$\mathbf{v}:$ = focus_vector
$\mathbf{f} = 0.5 * d * $ cot_theta

Ellipse: $\cos\theta > \sin\alpha$

Recalling the assumptions stated in Equation 1, we see clearly from Equations 5 and 6 that both r_1 and r_2 are positive with $r_1 > r_2$, and that the tangent spheres lie on opposite sides of the plane (see Figure 9). We can therefore express the foci of the ellipse as

$$F_1 = V + \frac{r_1}{\sin\alpha}\mathbf{w}_{cone} - r_1\mathbf{w}_{plane}$$

$$F_2 = V + \frac{r_2}{\sin\alpha}\mathbf{w}_{cone} + r_2\mathbf{w}_{plane}$$

The center is the midpoint of the foci:

$$C = \frac{F_1 + F_2}{2} = V + \frac{r_1 + r_2}{2\sin\alpha}\mathbf{w}_{cone} - \frac{r_1 - r_2}{2}\mathbf{w}_{plane}$$

But from Equations 5 and 6, we get

$$r_1 + r_2 = \frac{2((V - B) \cdot \mathbf{w}_{plane})\sin\alpha\cos\theta}{\sin^2\alpha - \cos^2\theta} = 2h\sin\alpha\cos\theta \quad (8)$$

and

$$r_1 - r_2 = \frac{2((V - B) \cdot \mathbf{w}_{plane})\sin^2\alpha}{\sin^2\alpha - \cos^2\theta} = 2h\sin^2\alpha \quad (9)$$

where

$$h = \frac{t}{b}$$
$$t = (B - V) \cdot \mathbf{w}_{plane}$$
$$b = \cos^2\theta - \sin^2\alpha$$

Therefore the center of the ellipse is expressed as

$$C = V + h\cos\theta\mathbf{w}_{cone} - h\sin^2\alpha\mathbf{w}_{plane}$$

The major axis vector $\mathbf{u}$ is parallel to $(F_1 - F_2)$. However, it really depends only on the relative orientation of the plane and cone. This is fortunate, since calculating $\mathbf{u}$ as normalize $(F_1 - F_2)$ involves considerable computation and will likely be numerically unstable as the foci get very close. Since we are interested only in the direction of $\mathbf{u}$, it suffices to consider any scalar multiple of $(F_1 - F_2)$.

$$\frac{1}{r_2}(F_1 - F_2) = \left(\frac{\frac{r_1}{r_2} - 1}{\sin\alpha}\right)\mathbf{w}_{cone} - \left(\frac{r_1}{r_2} + 1\right)\mathbf{w}_{plane} \quad (11)$$

Thus we need only the ratio r_1/r_2. From Equations 5 and 6,

$$\frac{r_1}{r_2} = \frac{\cos\theta + \sin\alpha}{\cos\theta - \sin\alpha} \qquad (12)$$

Substituting Equation 12 into Equation 11, we write

$$\frac{1}{r_2}(F_1 - F_2) = \left(\frac{2\sin\alpha}{\sin\alpha(\cos\theta - \sin\alpha)}\right)\mathbf{w}_{\text{cone}} - \left(\frac{2\cos\theta}{\cos\theta - \sin\alpha}\right)\mathbf{w}_{\text{plane}}$$

Again, since any scalar multiple of $(F_1 - F_2)$ suffices, we can conclude that $\mathbf{u}$ is parallel to

$$\frac{\cos\theta - \sin\alpha}{2r_2}(F_1 - F_2) = \mathbf{w}_{\text{cone}} - \cos\theta\mathbf{w}_{\text{plane}}$$

The direction of the major axis vector is therefore the component of $\mathbf{w}_{\text{cone}}$ perpendicular to $\mathbf{w}_{\text{plane}}$. The direction of the minor axis vector is perpendicular to the major axis vector and the plane normal vector. Therefore, as indicated in the algorithm below, we can easily compute it by a cross product.

All that remains is to find the major and minor radii. But from the Tangent Ball Theorem, we know that the distance along a cone ruling from one tangent circle to the other is twice the major radius. Hence,

$$r_u = \frac{1}{2}\left(\frac{r_1}{\tan\alpha} - \frac{r_2}{\tan\alpha}\right) = \frac{r_1 - r_2}{2\tan\alpha} = h\sin\alpha\cos\alpha \qquad (13)$$

Let d be the distance between the center and the focus of the ellipse. Then we can compute the minor radius as

$$r_v = \sqrt{r_u^2 - d^2} \qquad (14)$$

Since d is also half the distance between the foci of the ellipse, we can write

$$d^2 = \frac{1}{4}(F_1 - F_2)\cdot(F_1 - F_2)$$

$$= \frac{1}{4}\left\{\left(\frac{r_1 - r_2}{\sin\alpha}\right)^2 - 2\cos\theta\frac{(r_1 - r_2)(r_1 + r_2)}{\sin\alpha} + (r_1 + r_2)^2\right\} \qquad (15)$$

Substituting Equations 8 and 9 into Equation 15, we get

$$d^2 = \frac{1}{4}\{(2h\sin\alpha)^2 - 2\cos\theta(2h\sin^2\alpha)(2h\cos\theta)$$
$$+ (2h\sin\alpha\cos\theta)^2\}$$
$$= h^2\sin^2\alpha(1 - 2\cos^2\theta + \cos^2\theta)$$
$$= h^2\sin^2\alpha\sin^2\theta \qquad (16)$$

Substituting Equations 13 and 16 into Equation 14, we get

$$r_v = \sqrt{h^2\sin^2\alpha\cos^2\alpha - h^2\sin^2\alpha\sin^2\theta}$$
$$= h\sin\alpha\sqrt{\cos^2\alpha - \sin^2\theta}$$
$$= h\sin\alpha\sqrt{\cos^2\theta - \sin^2\alpha}$$

Finally, using the definition of h in Equation 10, we write

$$r_v = \frac{t\sin\alpha\sqrt{\cos^2\theta - \sin^2\alpha}}{\cos^2\theta - \sin^2\alpha} = \frac{t\sin\alpha}{\sqrt{b}}$$

We summarize these results in the following algorithm. Again we assume that the conditions in Equation 1 are satisfied. Though our derivation was again long and tedious, note how compact the final code becomes.

input: P: plane; C: cone
cos_theta : = C.**w** · P.**w**
cos_sqr_theta : = cos_theta * cos_theta
sin_alpha : = sin(C.α)
sin_sqr_alpha : = sin_alpha * sin_alpha
cos_alpha : = sqrt(1.0 – sin_sqr_alpha)
t : = (P.B – C.V) · P.**w**
b : = cos_sqr_theta – sin_sqr_alpha
h : = t / b
output: ellipse:
    C: = C.V + h * cos_theta * C.**w** – h * sin_sqr_alpha * P.**w**
    u: = normalize(C.**w** – cos_theta * P.**w**)
    v: = P.**w** × **u**
    ru: = h * sin_alpha * cos_alpha
    rv: = t * sin_alpha / sqrt(b)

If the plane normal and cone axis vector are parallel (that is, if $\mathbf{w}_{\text{cone}} \cdot \mathbf{w}_{\text{plane}} = 1$), the plane section is a circle. The discussion under "Plane-cylinder intersection" with respect to detecting after the fact when an ellipse is actually a circle applies here as well. We therefore test explicitly for this special case. We use the following algorithm if $\cos\theta$ as calculated above is determined to be 1 (see Figure 10).

input: P: plane; C: cone
h : = signed_distance_from_plane(C.V,P)
output: circle: C: = C.V – h * P.**w**
    w: = P.**w**
    r: = abs(h) * tan(C.α)

Hyperbola: $\cos\theta < \sin\alpha$

By using a slight variation of the algorithm we just developed for the ellipse, we can handle the hyperbola and the ellipse with a common algorithm.

Recalling the assumptions stated in Equation 1, we can clearly see from Equations 5 and 6 that r_1 is negative and r_2 is positive. The tangent spheres in this case lie on the same side of the plane (see Figure 11). We can therefore construct the foci of the hyperbola as

$$F_1 = V + \frac{r_1}{\sin\alpha}\mathbf{w}_{\text{cone}} - r_1\mathbf{w}_{\text{plane}}$$
$$F_2 = V + \frac{r_2}{\sin\alpha}\mathbf{w}_{\text{cone}} + r_2\mathbf{w}_{\text{plane}}$$

Notice that these are the same formulas as those derived for the ellipse. Therefore we can use the same formulas for the center and axis directions as we did for the ellipse. All that remains is to find the major and minor radii. We write the major
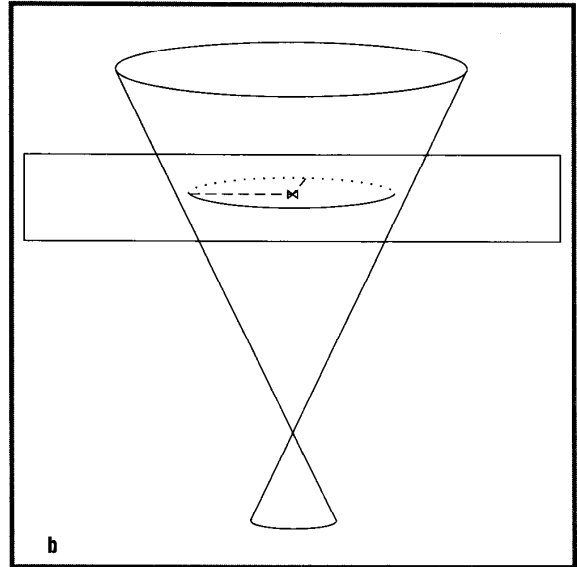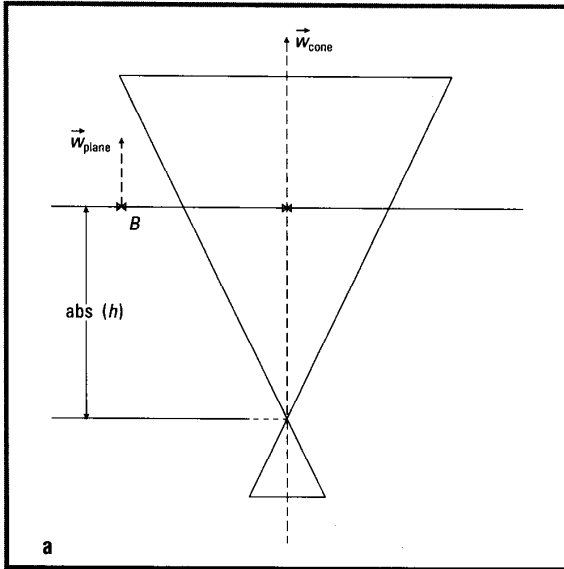
Figure 10. (a) The plane intersects the cone in a circle. (b) Off-axis view of the geometry of 10a.
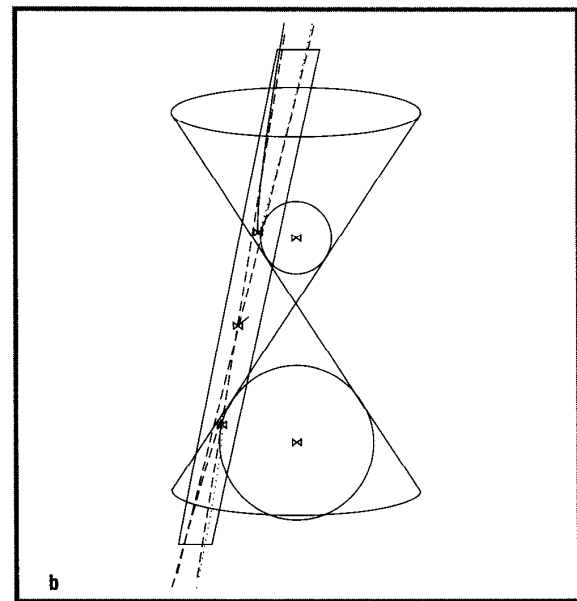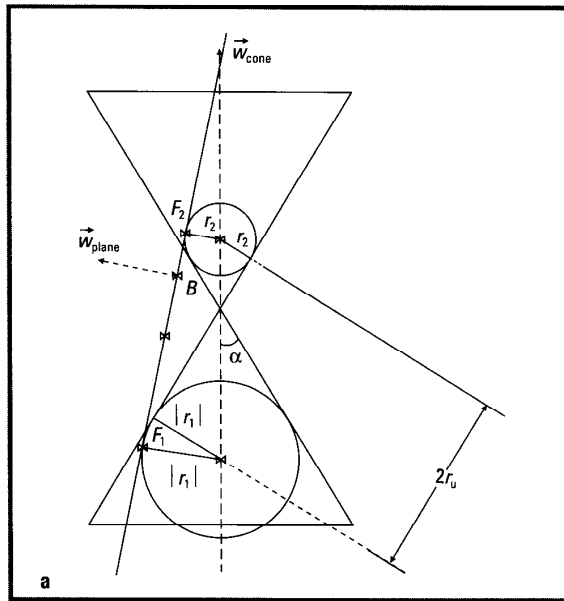

Figure 11. (a) The plane intersects the cone in a hyperbola $r_1 < 0$. (b) Off-axis view of the geometry of 11a.

radius as half the distance along a cone ruling from one tangent circle to the other:

$$r_u = \frac{1}{2}\left(\frac{r_2}{\tan \alpha} - \frac{r_1}{\tan \alpha}\right) = \frac{r_2 - r_1}{2 \tan \alpha}$$

This differs from the formula for r_u derived for the ellipse since here we have $(r_2 - r_1)$ instead of $(r_1 - r_2)$. Since $r_1 > r_2$ for the ellipse whereas $r_2 > r_1$ for the hyperbola, both formulas describe positive numbers.

Now if d is the distance between the center and the focus of the hyperbola, the minor radius is

$$r_v = \sqrt{d^2 - r_u^2}$$

This differs from the formula for r_v derived for the ellipse since here we have $(d^2 - r_u^2)$ instead of $(r_u^2 - d^2)$. For both the ellipse and the hyperbola, we consider a right triangle whose sides have lengths r_u, r_v, and d. The hypotenuse in the ellipse has length r_u, while that for the hyperbola has length d. Therefore, again both formulas describe positive numbers.

Exploiting these observations, we add two absolute value operations to the ellipse algorithm (one in the computation of r_u and the other in the computation of r_v) to get a common algorithm for both the ellipse and the hyperbola. Again we assume that the conditions stated in Equation 1 are satisfied.

input: P: plane; C: cone
cos_theta : = C.**w** · P.**w**
cos_sqr_theta : = cos_theta * cos_theta
sin_alpha : = sin(C.α)
sin_sqr_alpha : = sin_alpha * sin_alpha
cos_alpha : = sqrt(1.0 – sin_sqr_alpha)
t : = (P.B – C.V) · P.**w**
b : = cos_sqr_theta – sin_sqr_alpha
h : = t / b
output: ellipse (if cos_theta > sin_alpha) or hyperbola
      (if cos_theta < sin_alpha):
    C: = C.V + h * cos_theta * C.**w** – h * sin_sqr_alpha * P.**w**
    u: = normalize(C.**w** – cos_theta * P.**w**)
    v: = P.**w** × **u**
    ru: = abs(h) * sin_alpha * cos_alpha
    rv: = t * sin_alpha / sqrt(abs(b))

When the cone vertex is on the plane

If the vertex of the cone is on the plane, then the intersection is either a single point (if $\cos \theta > \sin \alpha$), one line (if $\cos \theta = \sin \alpha$), or two lines (if $\cos \theta < \alpha$). We cannot employ the tangent ball approach directly since in this case no spheres are tangent to both the plane and the cone. However, since lines are possible only when $\cos \theta \le \sin \alpha$, we can proceed by considering a limiting case of hyperbola intersections.

Consider a plane P' parallel to P at a distance ε from P, and suppose P' intersects the cone in a hyperbola. The asymptotes of the hyperbola are Line(C, $\mathbf{w}_1$) and Line(C, $\mathbf{w}_2$). The point C is the center of the hyperbola as computed by the algorithm described in the previous section. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are defined by $\mathbf{w}_1 = \text{Normalize}(\mathbf{u} + (r_v/r_u)\mathbf{v})$ and $w_2 = \text{Normalize}(\mathbf{u} - (r_v/r_u)\mathbf{v})$ where $\mathbf{u}$, $\mathbf{v}$, r_u, and r_v are the remaining geometric parameters of the hyperbola. In the limit as $\varepsilon \to 0$, the asymptotes of the hyperbola converge to the lines of intersection of P with the cone, and the center C of the hyperbola converges to the vertex V of the cone. We discovered in the previous section that the axis vectors $\mathbf{u}$ and $\mathbf{v}$ of the conic depend only on the relative orientation of the plane normal and the cone axis vectors; $\mathbf{u}$ and $\mathbf{v}$ did not depend on the position of the vertex relative to the plane. Therefore we need only find a formula for the

limit of the ratio (r_v/r_u) that also depends only on this relative orientation. Such a formula exists. since both r_u and r_v depend linearly on the distance of the vertex from the plane. From our earlier analysis. we know that when the intersection is a hyperbola,

$$r_u = \frac{((B - V) \cdot \mathbf{w}_{\text{plane}}) \sin \alpha \cos \alpha}{\sin^2 \alpha - \cos^2 \theta}$$

$$r_v = \frac{((B - V) \cdot \mathbf{w}_{\text{plane}}) \sin \alpha}{\sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

Therefore the ratio we seek is

$$\frac{r_v}{r_u} = \frac{\sqrt{\sin^2 \alpha - \cos^2 \theta}}{\cos \alpha} = \sqrt{\frac{\sin^2 \alpha - \cos^2 \theta}{1 - \sin^2 \alpha}} \tag{17}$$

Using this formula for the ratio, we can now express the lines of intersection when the vertex of the cone lies on the plane as

$$\text{Line}(V, \mathbf{u} + \frac{r_v}{r_u}\mathbf{v}) \quad \text{and} \quad \text{Line}(V, \mathbf{u} - \frac{r_v}{r_u}\mathbf{v}) \tag{18}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the hyperbola axis directions, and V is the vertex of the cone. When $\cos \theta > \sin \alpha$, we cannot compute the ratio in Equation 17. But since this is the case of a degenerate ellipse (described earlier), the intersection is just the vertex of the cone. When $\cos \theta = \sin \alpha$, the ratio in Equation 17 is zero. This is a degenerate parabola, and hence the intersection is a single tangent line whose direction is $\mathbf{u}$. Finally, when $\cos \theta < \sin \alpha$, the ratio in Equation 17 is a positive number. This is the degenerate hyperbola, and the intersection is the two lines described in Equation 18.

We summarize the handling of plane-cone intersections when the vertex of the cone is on the plane in the following algorithm. This algorithm doesn't require that $\mathbf{w}_{\text{cone}} \cdot \mathbf{w}_{\text{plane}}$ be nonnegative. Figure 12 illustrates the possible results.

input: P: plane; C: cone
cos_theta : = C.**w** · P.**w**
sin_alpha : = sin(C.α)
sin_sqr_alpha : = sin_alpha * sin_alpha
diff : = sin_sqr_alpha – cos_theta * cos_theta
if (diff < 0) **then**
  output: point: C.V
else
  u : = normalize(C.**w** – cos_theta * P.**w**)
  if (diff = 0) **then**
    output: line: B: = C.V
             w: = **u**
  else
    v : = P.**w** × **u**
    ratio : = sqrt(diff / (1-sin_sqr_alpha))
    output: line1: B: = C.V
              w: = normalize(**u** + ratio * **v**)
          line2: B: = C.V
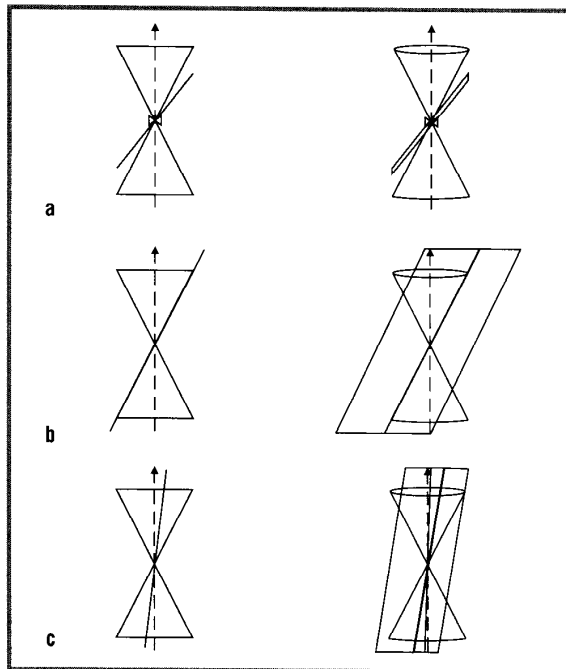   **w**: = normalize(**u** – ratio * **v**)

Figure 12. Intersection possibilities when the vertex lies on the sectioning plane: (a) vertex only, (b) single tangent line, and (c) two distinct lines.

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Conclusions

Using geometric constructions, we developed algorithms for computing plane sections of natural quadric surfaces. We present the plane and natural quadric to the algorithms in general position and orientation defined in terms of the geometric data shown in Table 2. The intersection (a conic, one or two lines, or a point) is returned in general position and orientation in terms of the geometric data shown in Table 1. We derived explicit formulas in terms of the plane and quadric parameters for all the geometric parameters of the conics. The algorithms operate entirely in world coordinates and do not employ coordinate system transformations of any sort.
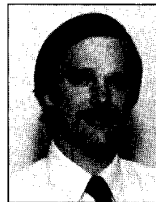
We implemented our algorithms in a solid modeling system being developed at the University of Kansas. Experience has shown them to be extremely reliable, efficient, and fast. ❏

Acknowledgment

James R. Miller is an associate professor of computer science at the University of Kansas. His research interests are computer graphics and geometrical modeling for mechanical CAD/CAM.

Miller received his BS in computer science from Iowa State University and his MS and PhD in computer science from Purdue University. He is a member of ACM Siggraph.
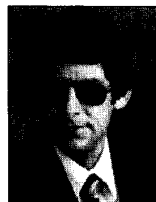
Ronald N. Goldman is a professor of computer science at Rice University. His research interests are computer-aided mathematical representation, manipulation, and shape analysis.

Goldman received a BS in mathematics from Massachusetts Institute of Technology and an MA and PhD in mathematics from Johns Hopkins University. He is a member of the editorial boards of *Computer-Aided Design* and *Computer-Aided Geometric Design.*

Miller can be reached at the University of Kansas, Dept. of Computer Science, 415 Snow Hall, Lawrence KS 66045-2192.