EECS 360 Short Quiz #3

Signal and System Analysis

October 25, 2012

Name: KEY

Open book, open notes, no calculator. Be neat, write legibly. For full credit you must show all work and justify each answer. You may write on the both sides of the paper, and use additional sheets of paper if needed. You may or may not find the following identity helpful:

$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right].$$

1. (25 %) Consider a causal LTI system with frequency response

$$H(j\omega) = \frac{1}{j\omega + 3}.$$

For a particular input $x(t)$ this system is observed to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t).$$

Determine $x(t)$.

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = h(t) * x(t)$$

$$X(j\omega) \qquad \underbrace{Y(j\omega) = H(j\omega)\cdot X(j\omega)}$$

$$\Rightarrow \boxed{X(j\omega) = \frac{Y(j\omega)}{H(j\omega)}}$$

$$Y(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega+4} = \frac{1}{(j\omega+3)(j\omega+4)}$$

$$H(j\omega) = \frac{1}{j\omega+3}$$

$$\Rightarrow X(j\omega) = \frac{1}{(j\omega+3)(j\omega+4)} \cdot (j\omega+3) \boxed{= \frac{1}{j\omega+4}}$$

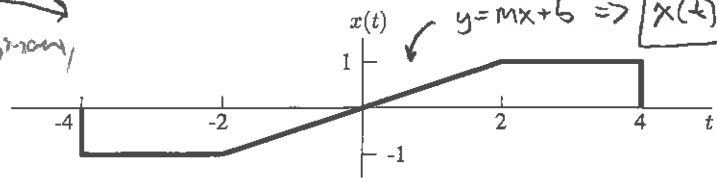
$$\Rightarrow \mathcal{F}^{-1}\left(\frac{1}{j\omega+4}\right) = \boxed{e^{-4t}u(t) = x(t)}$$

2. (40 %) Compute the continuous-time Fourier transform (CTFT) of the signal $x(t)$ plotted below.

$x(t)$ is real and odd

$X(f)$ is purely imaginary, and odd

$y = mx + b \Rightarrow \boxed{x(t) = \tfrac{1}{2}t}$



If, for whatever reason, you do not feel you can work the problem completely, then at least get some *partial credit* by 1) setting it up, and 2) telling me any CTFT properties you might use in arriving at the solution.

Solution # 1 : Plug & Chug

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int_{-4}^{4} x(t)e^{-j2\pi ft}\,dt = \underbrace{\int_{-4}^{-2}(-1)e^{-j2\pi ft}\,dt}_{\frac{-1}{-j2\pi f}e^{-j2\pi f}\Big|_{-4}^{2}\ (1)} + \underbrace{\int_{-2}^{2}\left(\tfrac{1}{2}t\right)e^{-j2\pi ft}\,dt}_{\text{Integration by parts}} + \underbrace{\int_{2}^{4}(1)e^{-j2\pi ft}\,dt}_{\frac{1}{-j2\pi f}e^{-j2\pi f}\Big|_{2}^{4}\ (2)}$$

$u = t \qquad dv = e^{-j2\pi ft}$

$du = dt \qquad v = \frac{-1}{j2\pi ft}e^{-j2\pi ft}$

$$\Rightarrow \frac{1}{2}\int_{-2}^{2} te^{-j2\pi ft}\,dt = \frac{t}{-j4\pi f}e^{-j2\pi ft}\Big|_{-2}^{2} - \frac{1}{2}\int_{-2}^{2}\frac{1}{-j2\pi f}e^{-j2\pi ft}\,dt$$

$$= \underbrace{\frac{-t}{j4\pi f}e^{-j2\pi ft}\Big|_{-2}^{2}}_{(3)} + \underbrace{\frac{1}{2(j2\pi f)^2}e^{-j2\pi ft}\Big|_{-2}^{2}}_{(4)}$$

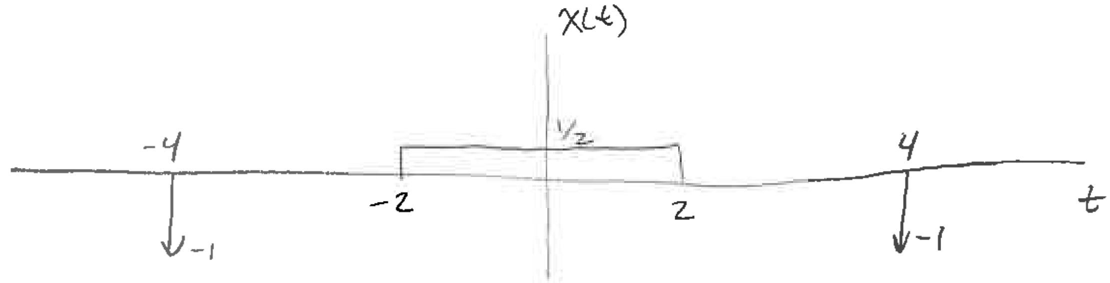
$$X(f) = \overbrace{\underbrace{\frac{1}{j2\pi f}e^{j4\pi f}}_{\text{cancel \#1}} - \frac{1}{j2\pi f}e^{j8\pi f}}^{(1)} + \overbrace{\frac{1}{-j2\pi f}e^{-j8\pi f} - \underbrace{\frac{1}{-j2\pi f}e^{-j4\pi f}}_{\text{cancel \#2}}}^{(2)} + \overbrace{\underbrace{\frac{-2}{j4\pi f}e^{-j4\pi f}}_{\text{cancel \#2}} - \underbrace{\frac{2}{j4\pi f}e^{j4\pi f}}_{\text{cancel \#1}}}^{(3)} \overbrace{- \frac{1}{-8\pi^2 f^2}e^{-j4\pi f} + \frac{1}{-8\pi^2 f^2}e^{j4\pi f}}^{(4)}$$

$$= \frac{-1}{j\pi f}\cos(8\pi f) + \frac{-j}{4\pi^2 f^2}\sin(4\pi f)$$

$$= \frac{j}{2\pi f}\left[2\cos(8\pi f) - \frac{1}{2\pi f}\sin(4\pi f)\right]$$

Solution # 2: Use the tables

Take the derivative of $x(t)$



$$-\delta(t+4) \overset{\mathcal{F}}{\longleftrightarrow} -e^{j8\pi f}$$

$$-\delta(t-4) \overset{\mathcal{F}}{\longleftrightarrow} -e^{-j8\pi f}$$

$$\rightarrow -2\cos(8\pi f)$$

$$\frac{1}{2}\,\text{rect}(t/4) \overset{\mathcal{F}}{\longleftrightarrow} 2\,\text{sinc}(4f) = \frac{\sin(4\pi f)}{2\pi f}$$

Apply integration property

$$\frac{1}{j2\pi f}\left[\frac{\sin(4\pi f)}{2\pi f} - 2\cos(8\pi f)\right] + \frac{1}{2}(2-2)\,\delta(f)$$

$$= \frac{j}{2\pi f}\left[2\cos(8\pi f) - \frac{\sin(4\pi f)}{2\pi f}\right]$$

Problem # 2, Solution # 3: Piecewise representation of $x(t)$

$$x(t) = -\text{rect}\left(\frac{t+3}{2}\right) + \frac{1}{2}t\,\text{rect}\left(\frac{t}{4}\right) + \text{rect}\left(\frac{t-3}{2}\right)$$

$$\Big\updownarrow \mathcal{F}$$

$$\underbrace{-2\,\text{sinc}(2f)\,e^{j2\pi\cdot 3f}}_{1^{st}\text{ Part of Euler's}} + j\frac{1}{4\pi}\frac{d}{df}4\,\text{sinc}(4f) + \underbrace{2\,\text{sinc}(2f)\,e^{-j2\pi\cdot 3f}}_{2^{nd}\text{ Part of Euler's}}$$

$$-j4\,\text{sinc}(2f)\sin(6\pi f) + \frac{j}{\pi}\frac{d}{df}\frac{\sin(4\pi f)}{4\pi f}$$

$$-\frac{j4}{2\pi f}\sin(2\pi f)\sin(6\pi f) + \frac{j}{\pi}\left[\frac{\cos(4\pi f)4\pi}{4\pi f} - \frac{\sin(4\pi f)4\pi}{(4\pi f)^2}\right]$$

apply identity $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

$$-\frac{j}{\pi f}\Big[\underbrace{\cos(4\pi f)}_{\text{cancel}} - \cos(8\pi f)\Big] + \frac{j}{\pi f}\left[\underbrace{\cos(4\pi f)}_{\text{cancel}} - \frac{\sin(4\pi f)}{4\pi f}\right]$$

$$X(f) = \frac{j}{2\pi f}\left[2\cos(8\pi f) - \frac{\sin(4\pi f)}{2\pi f}\right]$$

3. (35 %) Consider the following three continuous-time signals, each with a fundamental period of $T_0 = 1/2$:

$$x(t) = \cos(4\pi t),$$
$$y(t) = \sin(4\pi t),$$
$$z(t) = x(t)y(t).$$

$$\boxed{\text{We will use } T = T_0 = \tfrac{1}{2}}$$

(a) Determine the CTFS harmonic function $c_x[k]$ of $x(t)$.

(b) Determine the CTFS harmonic function $c_y[k]$ of $y(t)$.

(c) Use the results of parts (a) and (b), along with the multiplication property of the CTFS, to determine the CTFS harmonic function $c_z[k]$ of $z(t)$.

(d) Determine the CTFS harmonic function $c_z[k]$ of $z(t)$ through direct expansion of $z(t) = x(t)y(t)$ in trigonometric form, and compare your result with that of part (c).

(a) $x(t) = \cos(4\pi t) = \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} = \underset{\uparrow\ C_x[1]}{\frac{1}{2}e^{j2\pi t/T}} + \underset{\uparrow\ C_x[-1]}{\frac{1}{2}e^{-j2\pi t/T}}$

$\boxed{\text{By inspection } C_x[k] = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]}$

(b) $y(t) = \sin(4\pi t) = \underset{\uparrow\ C_y[1]}{\frac{1}{2j}e^{j2\pi t/T}} - \underset{\uparrow\ C_y[-1]}{\frac{1}{2j}e^{-j2\pi t/T}} \Rightarrow \boxed{C_y[k] = \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]}$

(c) Using the multiplication property

$z(t) = x(t) * y(t) \xleftrightarrow{FS} C_z[k] = C_x[k] * C_y[k]$

(FOIL)

$= \left[\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k-1]\right] * \left[\frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]\right]$

$= \underbrace{\frac{1}{4j}\delta[k-1]*\delta[k-1]}_{\text{First}} - \underbrace{\frac{1}{4j}\delta[k-1]*\delta[k+1]}_{\text{Outside}}$

$+ \underbrace{\frac{1}{4j}\delta[k-1]\delta[k+1]}_{\text{Inside}} - \underbrace{\frac{1}{4j}\delta[k+1]*\delta[k+1]}_{\text{Last}}$

$\boxed{= \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2]}$

(d) $z(t) = \cos(4\pi t)\sin(4\pi t) = \frac{1}{2}\left[\sin(4\pi t + 4\pi t) - \sin(4\pi t - 4\pi t)\right] = \frac{1}{2}\sin(8\pi t)$

$\frac{1}{2}\sin(8\pi t) = \underset{\uparrow\ C_z[2]}{\frac{1}{4j}e^{j2\pi(2)t/T}} - \underset{\uparrow\ C_z[-2]}{\frac{1}{4j}e^{j2\pi(-2)t/T}} \Rightarrow \boxed{C_z[k] = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2]}$