We will now prove the associativity property of convolution

(a) Prove $[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)]$ (1)

by showing that both sides of (1) equal

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)\,d\tau\,d\sigma$$

The left-hand side of (1) is

$$\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} x(\tau)h(\lambda-\tau)\,d\tau\right]}_{f(\lambda)} g(t-\lambda)\,d\lambda$$

Let $\lambda-\tau=\sigma \Rightarrow d\tau=-d\sigma \Rightarrow \lambda=\sigma+\tau$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)\,d\tau\,d\sigma$$ ← this is what we are looking for (2)

Now lets work with the right-hand side of (1)

$$\int_{-\infty}^{\infty} x(t-\lambda)\underbrace{\left[\int_{-\infty}^{\infty} h(\sigma)g(\lambda-\sigma)\,d\sigma\right]}_{f(\lambda)} d\lambda$$

Let $t-\lambda=\tau \Rightarrow \lambda=t-\tau,\ d\tau=-d\lambda$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)\,d\tau\,d\sigma$$ ← This is the same as (2)

So, since both sides of (1) are equal to (2), then (1) is true

[continued...]

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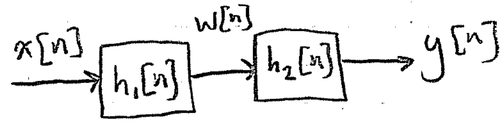
(b) Consider the cascaded system

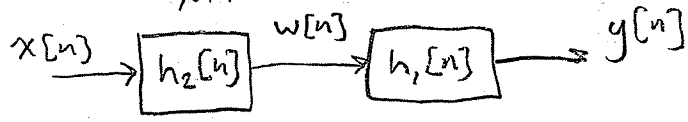


where $h_1[n] = \sin(8n)$ and $h_2[n] = a^n u[n]$, $|a| < 1$

and the input is $x[n] = \delta[n] - a\,\delta[n-1]$

Determine $y[n]$ (we will use the associative and commutative properties of convolution to make this easier)

Re-connect the system as



$$
\begin{aligned}
w[n] &= h_2[n] * \left(\delta[n] - a\,\delta[n-1]\right) \\
&= h_2[n] - a\,h_2[n-1] && \leftarrow \text{sifting, or sampling, property} \\
&= a^n u[n] - a\left(a^{n-1} u[n-1]\right) \\
&= a^n \left(u[n] - u[n-1]\right) && \leftarrow \delta[n] \text{ is } 1^{st} \text{ backward diff of } u[n] \\
&= a^n \delta[n] \\
&= \delta[n]
\end{aligned}
$$

$$
\begin{aligned}
y[n] &= \delta[n] * h_1[n] \\
&= h_1[n] \\
&= \sin(8n)
\end{aligned}
$$